

Geodesic complexity of the octahedron, and an algorithm for cut loci on convex polyhedra

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Abstract

The geodesic complexity of a length space X quantifies the required number of case distinctions to continuously choose a shortest path connecting any given start and end point. We prove a local lower bound for the geodesic complexity of X obtained by embedding simplices into $X \times X$. We additionally create and prove correctness of an algorithm to find cut loci on surfaces of convex polyhedra, as the structure of a space's cut loci is related to its geodesic complexity. We use these techniques to prove the geodesic complexity of the octahedron is four. Our method is inspired by earlier work of Recio-Mitter and Davis, and thus recovers their results on the geodesic complexity of the n -torus and the tetrahedron, respectively.

1 Introduction

In our work, we inspect the geodesic complexity of surfaces of polyhedra, taking inspiration from Davis's results on the tetrahedron [5] and cube [6]. We prove Theorem 2.1, which gives a general criterion for lower bounding the geodesic complexity of a length space X , witnessed by embedding simplices into $X \times X$. We additionally give an algorithm to calculate the cut locus of a point on a polyhedral surface. The methods refined and developed here in particular yield:

Theorem 1.1. *The geodesic complexity of the octahedron is four.*

1.1 Topological Complexity

Topological complexity was introduced by Farber [7] as a way to describe the difficulty of creating a continuous motion plan over a space X .

Definition 1.1. *The **free path space** PX is the space of paths $C([0, 1], X)$ imbued with the compact-open topology. This space comes with a **projection function** $\pi: PX \rightarrow X \times X$ via $\pi(\gamma) = (\gamma(0), \gamma(1))$.*

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A natural question to ask is if there is a local right inverse to π . Such a function σ would describe a continuous mapping from $X \times X$ (the space of start and end points in X , equipped with the product topology) into PX (the space of paths in X) such that $\sigma(x, y)$ is a path from x to y .

Definition 1.2. For $U \subseteq X \times X$, a **motion planning rule** is a map $\sigma: U \rightarrow PX$ such that $\pi \circ \sigma = \text{id}$.

A global motion planning rule on a space imposes strong topological constraints: The space must be contractible. Thus, topological complexity is defined through covering $X \times X$ with open sets that each have a local motion planning rule:

Definition 1.3. The **topological complexity** $TC(X)$ is the minimal k such that there exists open cover $U_1 \cup \dots \cup U_k = X \times X$ where each U_i has a motion planning rule $\sigma_i: U_i \rightarrow PX$.

1.2 Geodesic Complexity

For a metric space (X, d) , the length of a path $\gamma: [0, 1] \rightarrow X$ is

$$L(\gamma) = \sup_{0=t_0 \leq \dots < t_N=1} \left[\sum_{i=1}^N d(\gamma(t_{i-1}), \gamma(t_i)) \right].$$

Since we have a notion of length, a natural extension of topological complexity is to require the paths in each motion planning rule to be shortest paths. This was introduced by Recio-Mitter [15], and is constructed analogously to topological complexity.

Definition 1.4. A path γ is a **geodesic** if there exists $\lambda \in \mathbb{R}$ such that $\forall t, t'$ with $0 \leq t < t' \leq 1$,

$$d(\gamma(t), \gamma(t')) = \lambda |t - t'|.$$

This definition of a geodesic is ‘the shortest path with constant speed’. If a path has minimal length, it can always be reparameterized into a geodesic. Additionally, each path with minimal length has a unique reparameterization into a geodesic.

We will only consider **length spaces** [2], where the metric $d(x, y)$ is equal to the infimal path length of all paths between x and y : $d(x, y) = \inf_{\gamma \in \pi^{-1}(x, y)} L(\gamma)$.

Definition 1.5. Let $GX \subseteq PX$ be the subset containing only geodesics. We define a projection function $\pi_{GX} := \pi|_{GX}$ via the restriction of the projection function for PX .

Definition 1.6. Since $[0, 1]$ is compact and (X, d) is a metric space, GX is metrizable [11]. The metric we will use is $d_{GX}(f, g) = \sup_{t \in [0, 1]} d(f(t), g(t))$ for $f, g \in GX$.

Definition 1.7. For $E \subseteq X \times X$, a **Geodesic Motion Planning Rule (GMPR)** is a map $\sigma: E \rightarrow GX$ such that $\pi_{GX} \circ \sigma = \text{id}$. We say a set of geodesics $\mathcal{G}$ **extends** to a GMPR σ if $\forall G \in \mathcal{G}$, $\sigma(\pi_{GX}(G)) = G$.

Definition 1.8. The **geodesic complexity** $GC(X)$ of a length space X is the smallest k for which there is a partition into $k + 1$ locally compact sets $\bigsqcup_{i=0}^k E_i = X \times X$ with GMPRs $\sigma_i: E_i \rightarrow GX$. We call the collection of σ_i a **geodesic motion planner**.

We switch to covering $X \times X$ with locally compact sets out of necessity, as continuing to use an open cover would result in simple spaces such as S^1 having an undefined geodesic complexity (see [15, Remark 3.17]). Additionally we add one to the number of sets due to convention. To reduce confusion, we will clarify the number of sets along with any statement about geodesic complexity.

1.3 Document Structure

In Section 1, we introduce the concepts of topological and geodesic complexity. In Section 2, we develop methods we use to bound geodesic complexity. In Section 3, we apply our methods to produce novel results, as well as to reprove existing bounds in the literature.

2 Methods

2.1 Geodesic complexity? It's actually quite simplex

We will prove a local obstruction that bounds geodesic complexity of a space X from below by the dimension of certain simplices we can embed into $X \times X$. This is inspired by lower bound arguments of Davis [5, 6, Thm. 4.1] and Recio-Mitter [15, Thm. 4.4], whose proofs utilize sequences, sequences of sequences, and so on of elements in $X \times X$. Their proofs construct each successive iteration so that it cannot be in a GMPR with previous points. This shows that the geodesic complexity of X must be at least the number of iterations in this procedure. We adapt this by replacing n -dimensional sequences with embeddings of Δ_n into $X \times X$. Our hypotheses will let us use the same recursive sequence argument on these simplices, packaging all the machinery to show a lower bound from a collection of embeddings. Moving forward, we utilize this to lower bound geodesic complexity without the need to reference iterations of sequences.

Definition 2.1. The **affine hull** $\text{aff}(S)$ of a set $S \subseteq \mathbb{R}^n$ is the space of all affine combinations of S :

$$\text{aff}(S) = \left\{ \sum_{i=1}^k \alpha_i x_i \mid k > 0, x_i \in S, \alpha_i \in \mathbb{R}, \sum_{i=1}^k \alpha_i = 1 \right\}.$$

Definition 2.2. The **relative interior** $\text{relint}(S)$ for $S \subseteq \mathbb{R}^n$ is the interior of S in the affine hull of S :

$$\text{relint}(S) = \{x \in S : \text{there exists } \epsilon > 0 \text{ such that } N_\epsilon(x) \cap \text{aff}(S) \subseteq S\}.$$

If we embed set $S \subseteq \mathbb{R}^n$ via $f: S \hookrightarrow X$, we denote $\text{relint}^*(f) = f(\text{relint}(S))$.

Theorem 2.1. For integers $0 \leq k_0 < k$, if for each $i \in \{k_0, k_0 + 1, \dots, k\}$ there exists a non-empty collection $\mathcal{F}_i$ of embeddings into $X \times X$ such that:

- (a) Each $\mathcal{F}_i$ consists of embeddings of i -dimensional simplices $\Delta_i \hookrightarrow X \times X$.
- (b) For $F \in \mathcal{F}_i$, any geodesic in $\pi_{GX}^{-1}(\text{relint}^*(F))$ extends to a GMPR on all of $\text{img}(F)$.
- (c) With $k_0 \leq i < k$, for any $F \in \mathcal{F}_i$ and GMPR Γ on $\text{img}(F)$, there is an $F' \in \mathcal{F}_{i+1}$ such that:
 - (i) The map F is the restriction of F' to a face of Δ_{i+1} . We describe this relation as F being a face of F' .
 - (ii) Each geodesic in $\Gamma(\text{img}(F))$ is separated from any geodesics in $\pi_{GX}^{-1}(\text{relint}^*(F'))$ by an open set, that is, $\Gamma(\text{img}(F))$ does not intersect $\overline{\pi_{GX}^{-1}(\text{relint}^*(F'))}$, the closure of all geodesics on $\text{relint}^*(F')$.

(iii) There are a finite number of GMPRs on $\text{img}(F')$.

Then $GC(X) \geq k - k_0$. (This space requires at least $k - k_0 + 1$ sets in any geodesic motion planner).

We consider this a local obstruction since most of these hypotheses will hold when decreasing the size of the simplex embeddings. This is trivially true for hypothesis (a) and (b). For hypothesis (c), if we take care in the way we decrease the simplex embedding size, hypotheses (i) and (ii) will still hold. We display an example on the 2-torus in Section 3.1, where the hypotheses hold at arbitrarily small scale.

Proof: Assume for the sake of contradiction that $GC(X) < k - k_0$. Then partition $X \times X$ into $k - k_0$ locally compact sets $E_{k_0}, \dots, E_{k-1}$ such that each E_i has a GMPR σ_i . We will proceed by inductively constructing $k - k_0 + 1$ collections of points, where each collection belongs to a unique E_i .

Base Case (Step k_0): Pick an embedding $F_{k_0} \in \mathcal{F}_{k_0}$, and let $(p_*, q_*) \in \text{relint}^*(F_{k_0})$. Without loss of generality let E_{k_0} be the set that contains (p_*, q_*) . Let Γ_{k_0} be a GMPR on $\text{img}(F_{k_0})$ that extends $\sigma_{k_0}(p_*, q_*)$. This is possible from assumption (b).

Induction Hypothesis (Step i with $k_0 < i \leq k$): Assume for each $j \in \{k_0 + 1, k_0 + 2, \dots, i - 1\}$, we have chosen an embedding $F_j \in \mathcal{F}_j$ and sequences $(p_{s'}, q_{s'})_{s' \in \mathbb{N}^{j-k_0}} \subseteq \text{relint}^*(F_j)$ such that:

- (1)_{emb}^(j): F_{j-1} is a face of F_j .
- (2)_{emb}^(j): For all $j' \in \{k_0, \dots, j - 1\}$, any geodesic in $\Gamma_{j'}(\text{img}(F_{j'}))$ is some distance δ from all geodesics in $\pi_{GX}^{-1}(\text{relint}^*(F_j))$. Equivalently, $\Gamma_{j'}(\text{img}(F_{j'})) \cap \overline{\pi_{GX}^{-1}(\text{relint}^*(F_j))}$ is empty.
- (3)_{emb}^(j): Any geodesic in $\pi_{GX}^{-1}(\text{relint}^*(F_j))$ extends to a GMPR on $\text{img}(F_j)$.
- (4)_{emb}^(j): There are a finite number of GMPRs on $\text{img}(F_j)$.

- (1)_{seq}^(j): $d((p_s, q_s), (p_{s,n}, q_{s,n})) \leq \frac{1}{n}$ for $s \in \mathbb{N}^{(j-k_0)-1}$. In the 1-dimensional case, $d((p_*, q_*), (p_n, q_n)) \leq \frac{1}{n}$.
- (2)_{seq}^(j): The points $\{(p_{s'}, q_{s'}) : s' \in \mathbb{N}^{j-k_0}\} \subseteq E_j$.
- (3)_{seq}^(j): There is a GMPR Γ_j on $\text{img}(F_j)$ that extends $\{\sigma_j(p_{s'}, q_{s'}) : s' \in \mathbb{N}^{j-k_0}\}$.
- (4)_{seq}^(j): All of these properties still hold under taking subsequences.

This hypothesis is trivially true for $i = k_0 + 1$, as there is no integer j such that $k_0 < j < k_0 + 1$.

Induction Step (Step i with $k_0 < i \leq k$): From assumptions (c) and (b), we can find an embedding $F_i \in \mathcal{F}_i$ such that:

- (1)_{emb}^{(i)*}: F_{i-1} is a face of F_i .
- (2)_{emb}^{(i)*}: For $j = i - 1$, the intersection $\Gamma_j(\text{img}(F_j)) \cap \overline{\pi_{GX}^{-1}(\text{relint}^*(F_i))}$ is empty.
- (3)_{emb}^{(i)*}: Any geodesic in $\pi_{GX}^{-1}(\text{relint}^*(F_i))$ extends to a GMPR on $\text{img}(F_i)$.
- (4)_{emb}^{(i)*}: There are a finite number of GMPRs on $\text{img}(F_i)$.

To show (1)_{emb}⁽ⁱ⁾, $\dots$, (4)_{emb}⁽ⁱ⁾, it is enough to show that (2)_{emb}^{(i)*} holds for all $j \in \{k_0, \dots, i - 1\}$.

Assume for the sake of contradiction that this were not true for some $k_0 \leq j < i - 1$ and geodesic $G \in \Gamma_j(\text{img}(F_j)) \cap \overline{\pi_{GX}^{-1}(\text{relint}^*(F_i))}$. Then $G = \Gamma_j(p, q)$ for $(p, q) \in \text{img}(F_j)$, and we can find for each $\delta \in (\frac{1}{n})_n$, a geodesic in $\pi_{GX}^{-1}(\text{relint}^*(F_i))$ that is δ -close to G . By (3)_{emb}^{(i)*}, each of these geodesics extend to a GMPR across $\text{img}(F_i)$. By (4)_{emb}^{(i)*} and pigeonhole principle, we can find a GMPR Γ on $\text{img}(F_i)$ that agrees with an infinite subsequence of these geodesics. Then for any distance δ , there exists $(p', q') \in \text{relint}^*(F_i)$

such that $\Gamma(p', q')$ is δ -close to G . By Definition 1.6, this implies (p, q) and (p', q') are (2δ) -close. Thus, by choosing $\delta \in (\frac{1}{n})_n$, we find a sequence $(p'_n, q'_n)_n$ that approaches (p, q) while $\Gamma(p'_n, q'_n)_n$ approaches G . Since GX is Hausdorff, this shows $\Gamma(p, q) = G$. However, since Γ is a GMPR restricted to $\text{img}(F_{j+1})$, we contradict $(2)_{\text{emb}}^{(j+1)}$, which asserts G is some distance from all geodesics in $\pi_{GX}^{-1}(\text{relint}^*(F_{j+1}))$, in particular, those in $\Gamma(\text{relint}^*(F_{j+1}))$.

Thus, we establish $(1)_{\text{emb}}^{(i)}, \dots, (4)_{\text{emb}}^{(i)}$. Next, for each $\mathbf{s} \in \mathbb{N}^{(i-k_0)-1}$, we can create a sequence¹ $(p_{\mathbf{s},n}, q_{\mathbf{s},n})_n \subseteq \text{relint}^*(F_i)$ such that:

$$(1)_{\text{seq}}^{(i)*}: d((p_{\mathbf{s}}, q_{\mathbf{s}}), (p_{\mathbf{s},n}, q_{\mathbf{s},n})) \leq \frac{1}{n}.$$

This is possible, as $(p_{\mathbf{s}}, q_{\mathbf{s}}) \in \text{relint}^*(F_{i-1})$, and $\text{relint}^*(F_{i-1}) \subseteq \overline{\text{relint}^*(F_i)}$ by $(1)_{\text{emb}}^{(i)}$.

$(2)_{\text{seq}}^{(i)*}$: The points $\{(p_{\mathbf{s}'}, q_{\mathbf{s}'}) : \mathbf{s}' \in \mathbb{N}^{i-k_0}\} \subseteq E_\ell$ for some E_ℓ . We show this is possible through induction: First, each point is trivially in a single set, showing our base case. Fix some $\mathbf{t} \in \mathbb{N}^{j-k_0}$ for $k_0 \leq j < i$. We inductively assume for each $n \in \mathbb{N}$, $\{(p_{\mathbf{t},n}, q_{\mathbf{t},n}) : (\mathbf{t}, n, \mathbf{t}') \in \mathbb{N}^{i-k_0}\} \subseteq E_{\ell_n}$. However, since there are finitely many choices for E_{ℓ_n} , one of these sets (say E_ℓ) occurs infinitely many times in the sequence $(E_{\ell_n})_n$. This corresponds to an infinite set $M \subseteq \mathbb{N}$ such that $\{(p_{\mathbf{t},m}, q_{\mathbf{t},m}) : (\mathbf{t}, m, \mathbf{t}') \in \mathbb{N}^{i-k_0}, m \in M\} \subseteq E_\ell$. By $(4)_{\text{seq}}^{(j+1)}$, we can assume without loss of generality that this subsequence $(p_{\mathbf{t},m}, q_{\mathbf{t},m})_{m \in M}$ is the whole sequence $(p_{\mathbf{t},n}, q_{\mathbf{t},n})_n$.

$(3)_{\text{seq}}^{(i)*}$: There is a GMPR Γ_i on $\text{img}(F_i)$ that extends $\{\sigma_\ell(p_{\mathbf{s}'}, q_{\mathbf{s}'}) : \mathbf{s}' \in \mathbb{N}^{i-k_0}\}$.

This can be proved possible similarly to $(2)_{\text{seq}}^{(i)*}$. The key observation is that using $(3)_{\text{emb}}^{(i)}$, we may associate each $\sigma_\ell(p_{\mathbf{s}'}, q_{\mathbf{s}'})$ (for $\mathbf{s}' \in \mathbb{N}^{i-k_0}$) with a GMPR across $\text{img}(F_i)$. By $(4)_{\text{emb}}^{(i)}$, there are finitely many of these, which allows us to use the same machinery as before.

$(4)_{\text{seq}}^{(i)*}$: All of these properties still hold under taking subsequences.

To show $(1)_{\text{seq}}^{(i)}, \dots, (4)_{\text{seq}}^{(i)}$, it is enough to show that $E_\ell \neq E_j$ for $j < i$.

Assume for the sake of contradiction that $E_\ell = E_j$ for some $j < i$. Then take some $(p_{\mathbf{r}}, q_{\mathbf{r}})$ for $\mathbf{r} \in \mathbb{N}^{j-k_0}$ (take $(p_{\mathbf{r}}, q_{\mathbf{r}}) = (p_*, q_*)$ if $j = k_0$). Consider the sequence $(p_{\mathbf{r},n}, q_{\mathbf{r},n})_n$ (where $(\mathbf{r}, n, \dots, n) \in \mathbb{N}^{i-k_0}$). By $(1)_{\text{seq}}^{(i)*}, (1)_{\text{seq}}^{(i-1)}, \dots, (1)_{\text{seq}}^{(j+1)}$, we have that $(p_{\mathbf{r},n}, q_{\mathbf{r},n})_n$ approaches $(p_{\mathbf{r}}, q_{\mathbf{r}})$. Through continuity of $\sigma_j = \sigma_\ell$, we construct a sequence $\sigma_j(p_{\mathbf{r},n}, q_{\mathbf{r},n})_n$ of geodesics approaching $\sigma_j(p_{\mathbf{r}}, q_{\mathbf{r}})$. However, this is a contradiction with $(2)_{\text{emb}}^{(i)}$, which claims since $(p_{\mathbf{r},n}, q_{\mathbf{r},n})_n \subseteq \text{relint}^*(F_i)$, there is an open set separating $\sigma_j(p_{\mathbf{r}}, q_{\mathbf{r}}) = \Gamma_j(p_{\mathbf{r}}, q_{\mathbf{r}})$ from all $\sigma_j(p_{\mathbf{r},n}, q_{\mathbf{r},n})_n$. Thus, $E_\ell \neq E_j$ for all $j < i$, and without loss of generality we can set $E_\ell = E_i$.

We can continue until $i = k$, creating distinct sets $E_{k_0}, \dots, E_k$ and contradicting the number of sets assumed. Therefore, $GC(X) \geq k - k_0$, needing at least $k - k_0 + 1$ sets in any geodesic motion planner. ■

We illustrate the mechanism behind the proof with an example where X is the 2-torus, showing $GC(X) \geq 2$. For collections of embeddings, let $\mathcal{F}_0 = \{\Delta_0 \mapsto (p_*, q_*)\}$ for some point p_* and its antipode q_* . Let $\varepsilon > 0$ be small. There are four axis-aligned lines on X of length ε that end in q_* (displayed as solid gray lines in Figure 1(a)). Let $\mathcal{F}_1$ consist of four maps $\Delta_1 \hookrightarrow X \times X$ with their first factor fixed at p_* and their second factor one of those four lines. Similarly, there are four axis-aligned squares on X with side length ε with a vertex at q_* (displayed as gray squares in Figure 1(a)). Let $\mathcal{F}_2$ consist of four maps $\Delta_2 \hookrightarrow X \times X$ with their first factor fixed at p_* and their second factor one of these four squares. With this construction, for $F_1 \in \mathcal{F}_1$, $\text{img}(F_1)$ can be written $\{p_*\} \times F'_1$ for a line F'_1 . Similarly, for $F_2 \in \mathcal{F}_2$, $\text{img}(F_2) = \{p_*\} \times F'_2$ for a square F'_2 . We display F'_1 and F'_2 for particular $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ in Figure 1(b) and (c) respectively. In

¹In the 1-dimensional case, this is $(p_n, q_n)_n$

Section 3.1, we describe this construction in more detail and show it satisfies the hypotheses of Theorem 2.1.

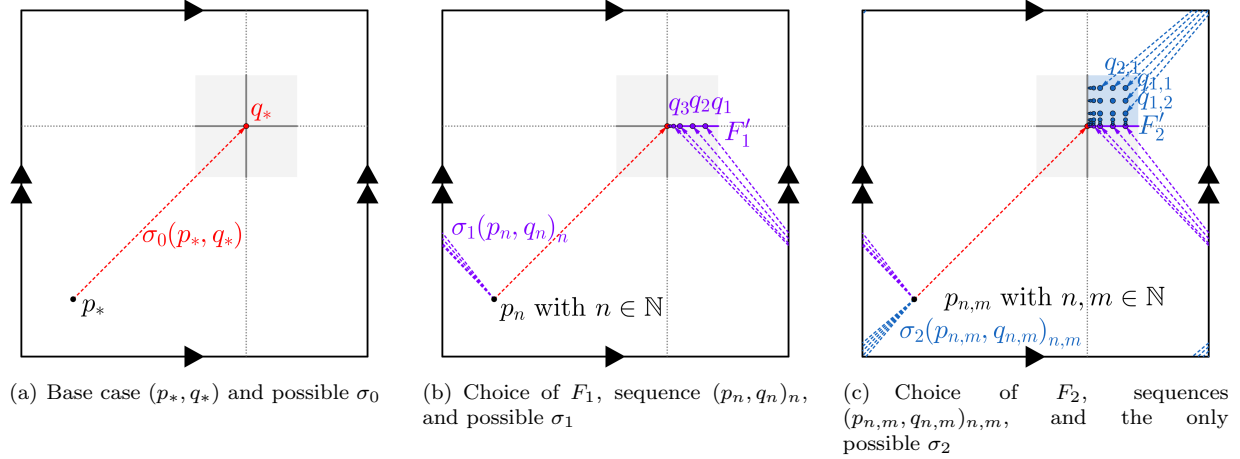


Figure 1: Mechanism of Theorem 2.1 when used on the 2-torus

We will follow along the construction in Theorem 2.1 for this explicit example. We begin by assuming for sake of contradiction that $GC(X) < 2$. We partition $X \times X$ into sets E_0 and E_1 with GMPRs σ_0 and σ_1 .

Base Case: We select $F_0 \in \mathcal{F}_0$, with $\text{img}(F_0) = (p_*, q_*)$. Without loss of generality we find $(p_*, q_*) \in E_0$, and find $\sigma_0(p_*, q_*)$ is as displayed in Figure 1(a). The geodesic $\sigma_0(p_*, q_*)$ extends to Γ_0 , a GMPR over $\text{img}(F_0)$.

Step 1: We select F_1 , an embedding in $\mathcal{F}_1$ where F_0 is a face of F_1 and $\Gamma_0(\text{img}(F_0)) \cap \overline{\pi_{GX}^{-1}(\text{relint}^*(F_1))}$ is empty. We choose sequence $(p_n, q_n)_n$ within $\text{relint}^*(F_1)$, where $\lim_{n \rightarrow \infty} (p_n, q_n) = (p_*, q_*)$. Without loss of generality we assume that $(p_n, q_n)_n \subseteq E_\ell$ and that $\sigma_\ell(p_n, q_n)_n$ is as displayed in Figure 1(b). Since $\sigma_\ell(p_n, q_n)_n$ does not approach $\sigma_0(p_*, q_*)$, this would yield a discontinuity if $\ell = 0$. Thus, $\ell = 1$. The geodesics $\{\sigma_1(p_n, q_n) : n \in \mathbb{N}\}$ extend to a GMPR Γ_1 over $\text{img}(F_1)$.

Step 2: We select F_2 , the only embedding in $\mathcal{F}_2$ where F_1 is a face of F_2 and $\Gamma_1(\text{img}(F_1)) \cap \overline{\pi_{GX}^{-1}(\text{relint}^*(F_2))}$ is empty. For each n , we choose sequence $(p_{n,m}, q_{n,m})_m$ within $\text{relint}^*(F_2)$, where $\lim_{m \rightarrow \infty} (p_{n,m}, q_{n,m}) = (p_n, q_n)$. Without loss of generality we assume that $(p_{n,m}, q_{n,m})_{m,n} \subseteq E_\ell$ and that $\sigma_\ell(p_{n,m}, q_{n,m})_{m,n}$ is as displayed in Figure 1(c). Since for each n , $\sigma_\ell(p_{n,m}, q_{n,m})_{m,n}$ does not approach $\sigma_1(p_n, q_n)$, this would yield a discontinuity if $\ell = 1$. Similarly, $\sigma_\ell(p_{k,k}, q_{k,k})_k$ does not approach $\sigma_0(p_*, q_*)$, and this would yield a discontinuity if $\ell = 0$. This is a contradiction, as σ_ℓ can only be continuous if ℓ is a unique value (say $\ell = 2$).

2.2 Embeddings of Simplices

Since Theorem 2.1 is concerned with embeddings of simplices into $X \times X$, we will describe a simple way to construct such embeddings in subsets of Euclidean spaces. The construction is motivated by the fact that after embedding the start point p into the first factor of $X \times X$, the embedding of the second factor can be chosen in a convex hull of points in X .

Definition 2.3. Let $T_k := \left\{ r \in \mathbb{R}^k : \sum_{i=1}^k r_i \leq 1; \forall i, r_i \geq 0 \right\}$, and let $S_k := \left\{ r \in T_k : \sum_{i=1}^k r_i = 1 \right\}$, which are k -dimensional and $(k-1)$ -dimensional simplices respectively. For $\mathbf{t} \in T_k$, $1 \leq k' \in \mathbb{N}$, let $D_{\mathbf{t},k'} := \{ \mathbf{x} \in \Delta_{k+k'-1} : (\mathbf{x}_1, \dots, \mathbf{x}_k) = \mathbf{t} \}$ be the subset of simplex $\Delta_{k+k'-1}$ whose first k coordinates are $\mathbf{t}$.

Definition 2.4. A finite set of points A in a Euclidean space is **affine independent** if no $a \in A$ can be written as $\sum_{a' \in A \setminus \{a\}} \lambda_{a'} a'$, where $\lambda_{a'} \in \mathbb{R}$ and $\sum_{a' \in A \setminus \{a\}} \lambda_{a'} = 1$. It is easily seen that if A is affine independent, its convex hull is homeomorphic to $\Delta_{|A|-1}$.

Lemma 2.1. For any space Z , subset Y of a Euclidean space, and integers $k, k' > 0$, let there be an embedding $p: T_k \hookrightarrow Z$ and a map $q: T_k \rightarrow Y^{k'}$. Assume that for $\mathbf{t} \in T_k \setminus S_k$, $q(\mathbf{t})$ maps to k' affinely independent points whose convex hull is contained within Y , and for $\mathbf{t} \in S_k$, $q(\mathbf{t}) = (y_{\mathbf{t}}, \dots, y_{\mathbf{t}})$ for some $y_{\mathbf{t}} \in Y$. Then there exists an embedding $f: \Delta_{k+k'-1} \hookrightarrow Z \times Y$ such that:

- For all $\mathbf{t} \in T_k$ and $\mathbf{x} \in D_{\mathbf{t},k'}$, the first factor $\pi_1(f(\mathbf{x})) = p(\mathbf{t})$.
- For all $\mathbf{t} \in T_k$, the image $\pi_2(f(D_{\mathbf{t},k'}))$ is the convex hull of the points defined by $q(\mathbf{t})$.

Proof: The map $q': T_k \rightarrow C(\Delta_{k'-1}, Y)$ can be constructed by mapping the i th vertex of $\Delta_{k'-1}$ to $q(\mathbf{t})_i$ for each $\mathbf{t} \in T_k$, and constructing the rest of $q'(\mathbf{t})$ linearly. The map $q'(\mathbf{t})$ is an embedding for $\mathbf{t} \in T_k \setminus S_k$ and maps all of $\Delta_{k'-1}$ to a point for $\mathbf{t} \in S_k$.

All elements of $\Delta_{k+k'-1}$ can be written $(\mathbf{t}, \mathbf{t}')$ for $\mathbf{t} \in T_k$. Using this, we define $f: \Delta_{k+k'-1} \rightarrow Z \times Y$:

$$f(\mathbf{t}, \mathbf{t}') = \begin{cases} \left(p(\mathbf{t}), q'(\mathbf{t}) \left(\frac{\mathbf{t}'}{\sum_i \mathbf{t}'_i} \right) \right) & \mathbf{t} \in T_k \setminus S_k \\ (p(\mathbf{t}), y) & \mathbf{t} \in S_k, \text{ and for } q'(\mathbf{t})(\Delta_{k'-1}) = \{(y, \dots, y)\} \end{cases}$$

Continuity follows from continuity of p and q . Since p is injective, and $\pi_1 \circ f(\mathbf{t}, \mathbf{t}') = p(\mathbf{t})$, it is sufficient to verify injectivity independently for each value of $\mathbf{t}$. For $\mathbf{t} \notin S_k$, injectivity of f follows from injectivity of $q'(\mathbf{t})$. For $\mathbf{t} \in S_k$, we have that $\mathbf{t}'$ can only be $(0, \dots, 0)$, so injectivity is trivial in this case. Thus, f is a well defined embedding $\Delta_{k+k'-1} \hookrightarrow Z \times Y$. The desired properties follow from construction. ■

To create a map $q: T_k \rightarrow Y^{k'}$, we will often combine $q_1, \dots, q_{k'}: T_k \rightarrow Y$ to create a ‘product,’ a map $T_k \rightarrow Y^{k'}$ via $\mathbf{t} \mapsto (q_1(\mathbf{t}), \dots, q_{k'}(\mathbf{t}))$. We will denote this map $q_1 \boxtimes \dots \boxtimes q_{k'}$.

2.3 Polyhedra

We inspect **convex polyhedra**, defined as the convex hull of finitely many points in $\mathbb{R}^3$. In particular, the spaces we are interested in are the 2-D surfaces of convex polyhedra, equipped with the **flat metric** that measures the length of the shortest path along the surface between points. These spaces may be equivalently viewed as the union of their polygonal faces, glued at appropriate edges. We will prove various lemmas for these spaces, and use them to prove correctness of an algorithm to calculate all geodesics from a point.

2.3.1 General Lemmas

Definition 2.5. Let X be a polyhedron, and consider the graph whose vertices are the polygon faces of X . Connect two faces if they share an edge. We will define a **face walk** as a walk in this graph, and a **face**

path as a path (a walk with no repeat faces). The graph considered is equivalent to the 1-skeleton of the dual polyhedron [19] of X .

Definition 2.6. Let $F_0, \dots, F_n$ be a face walk. A **walk unfolding** is a map $f: \bigcup F_i \rightarrow \mathbb{R}^2$ such that $f|_{F_i}$ is an isometric embedding for each F_i , and $f|_{F_i \cup F_{i+1}}$ is a homeomorphism for each $i < n$. We similarly define a **path unfolding** for face paths.

This is similar to the concept of an unfolding discussed in Shephard's Conjecture [18], with the main distinction being that in walk/path unfoldings, all faces of the polyhedron need not be represented. We display an example of a path unfolding obtained from an icosahedron in Figure 3(b).

Lemma 2.2. Given a face walk $F_0, \dots, F_n$ on a convex polyhedron and isometric embedding $g: F_i \hookrightarrow \mathbb{R}^2$, there is a unique walk unfolding f such that $f|_{F_i} = g$. We call this the walk unfolding with respect to g .

This can be shown through induction by considering the restriction of f to each face.

Lemma 2.3. Geodesics on a convex polyhedron cannot exit a face then revisit it.

Proof: Assume for the sake of contradiction a segment of geodesic G starts and ends on face F , but $\text{img}(G) \not\subseteq F$. We can without loss of generality assume G is this segment, with end points a and b . Consider a natural² embedding i of the whole polyhedron into $\mathbb{R}^3$ such that each face is isometrically embedded. It is clear that both i and its inverse preserve the lengths of paths, as the restriction to each face is an isometry. Thus, it is sufficient to find a path between $i(a)$ and $i(b)$ that is shorter than $i \circ G$, and is contained in the image of i . The 'straight line' path G' between $i(a)$ and $i(b)$ is contained in the image of F by convexity of faces, and thus is a valid path. We have that $i \circ G$ is distinct from G' , as $i \circ G([0, 1]) \not\subseteq \text{img}(F)$. This implies $i \circ G$ is longer than G' as in $\mathbb{R}^3$, any two points have a unique shortest path. Thus, $i^{-1} \circ G'$ is shorter than G , and we have a contradiction. ■

Lemma 2.4. Let G be a geodesic on convex polyhedron X . There exists a decomposition $G = G_0 \cup \dots \cup G_n$ where each G_i is a line segment or point that intersects G_{i+1} at a point, and $G_i \subseteq F_i$ for some face F_i of X . Additionally, $F_0, \dots, F_n$ is a face path, and the image of G is a line on the path unfolding of $F_0, \dots, F_n$.

Proof: To decompose G we first claim that the intersection of G with each edge of X must be connected, and thus either a point or a line segment. This may be shown similar to Lemma 2.3. We may then consider a particular edge e and split G into one or two segments based on a point $G \cap e$, or one to three segments based on the endpoints of line segment $G \cap e$. Continuing the decomposition on each segment for the remaining edges will create a finite ordered decomposition of non-point elements $G''_0, \dots, G''_{n''}$, where for any G''_i and any edge e of X , $G''_i \cap e$ is either empty, all G''_i , or a single endpoint of G''_i . Thus, each element must be contained in a face, as leaving a face would cause an intersection with an edge. Let $F''_0, \dots, F''_{n''}$ be these faces, where $G''_i \subseteq F''_i$. If the endpoint of G''_i (i.e. $G''_i \cap G''_{i+1}$) is on the interior of an edge of X , then F''_i and F''_{i+1} must share that edge. Otherwise, $G''_i \cap G''_{i+1}$ must be a vertex of X adjacent to F''_i and F''_{i+1} . In this case, all faces that share this vertex are joined in a cycle, and we may insert the appropriate faces in between F''_i and F''_{i+1} such that each adjacent pair share an edge. We will insert copies of the vertex $G''_i \cap G''_{i+1}$ to ensure the decomposition matches the list of faces. After doing this for all the original F''_i , we obtain a new decomposition $G'_0, \dots, G'_{n'}$, where $G'_i \subseteq F'_i$ and $F'_0, \dots, F'_{n'}$ is a face walk. Finally, if $F'_i = F'_j$ for $i < j$, $G'_i \cup \dots \cup G'_j$ is a continuous segment of a geodesic that begins and ends on the same face. Thus, by Lemma 2.3, this segment never leaves F'_i , and we may replace $F'_i, \dots, F'_j$ with just F'_i , and

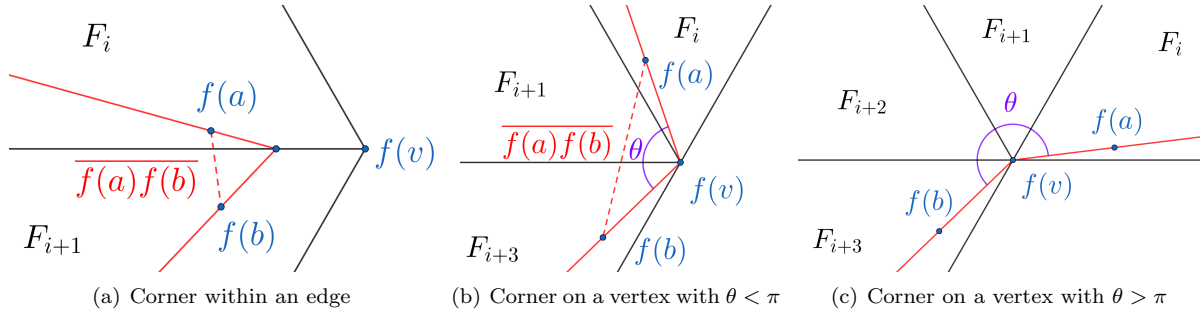


Figure 2: Contradictions arising from a corner within a geodesic

replace $G'_i, \dots, G'_j$ with the union $G'_i \cup \dots \cup G'_j$. Repeating this until no duplicates remain will create a face path $F_0, \dots, F_n$.

Consider a path unfolding f of $F_0, \dots, F_n$ onto $\mathbb{R}^2$, and consider the image of G wrt. this map. It is clear that each $f(G_i)$ must be a line segment or point, so $f(G)$ must be a union of line segments. We must show that the segments are collinear. If a ‘corner’ exists, it must either be on the edge or vertex of a face. In either case, we will reach a contradiction by creating a shorter path on a small neighborhood of the corner.

Assume for the sake of contradiction there is a corner on some edge. The image of $f|_{F_i \cup F_{i+1}}$ is two polygons that share an edge, and the assumed corner is on the interior of this edge. Thus, the corner is on the interior of the image of $f|_{F_i \cup F_{i+1}}$. Thus we can find a shorter path within the image between points $f(a)$ and $f(b)$ sufficiently close to the corner, visualized in Figure 2(a). Since both $f|_{F_i \cup F_{i+1}}$ and its inverse preserve the length of paths, we have a contradiction.

Now assume for the sake of contradiction there is a corner on a vertex v . Let ϵ be much smaller than the length of any edge containing v . Consider the section of the path that is ϵ -close to v , and let the endpoints be a and b . It is enough to show that the segment of G from a to b is not a geodesic. Let θ be the angle between $f(a), f(v)$, and $f(b)$, oriented as in Figure 2(b, c). If $\theta < \pi$, then we can create a shorter path by simply connecting the line $\overline{f(a)f(b)}$. We visualize this in Figure 2(b). If $\theta > \pi$ as in Figure 2(c), we will show we can choose a different face walk that reduces this to the case where $\theta < \pi$. All faces on the polyhedra connected to vertex v create a cycle. We will go the other way around the cycle (with associated walk unfolding g). Since polyhedra have positive curvature, the angle between $g(a), g(v)$, and $g(b)$ in this direction must be less than $2\pi - \theta$, and thus less than π . Then this reduces to the first case, and we can find a shorter path between a and b . ■

Definition 2.7. For a convex polyhedron X , face path $F_0, \dots, F_n$, and path unfolding f , let $p \in F_0$. We will say $q \in F_n$ has property $\star$ with respect to p if the line $P := \overline{f(p)f(q)}$ has the following properties:

- (a) $P \subseteq \text{img}(f)$.
- (b) The intersection $P \cap f(F_0)$ is a path from $f(p)$ to edge $f(F_0 \cap F_1)$, and the intersection $P \cap f(F_n)$ is a path from edge $f(F_{n-1} \cap F_n)$ to $f(q)$.
- (c) For $i \in \{1, \dots, n-1\}$, the intersection $P \cap f(F_i)$ is a path between edges $f(F_i \cap F_{i-1})$ and $f(F_i \cap F_{i+1})$.

Lemma 2.5. For convex polyhedron X , face path $F_0, \dots, F_n$, path unfolding f , and some $p \in F_0$, the set of $q \in F_n$ that satisfy property $\star$ (with respect to p) is convex.

²Polyhedra are defined as objects in $\mathbb{R}^3$, so there is a natural embedding, unique up to isometry.

Proof: Pick $q_1, q_2 \in F_n$ such that q_1 and q_1 have $\star$. Let $P_1 = \overline{f(p)f(q_1)}$ and $P_2 = \overline{f(p)f(q_2)}$. For some $\lambda \in (0, 1)$, let $q = \lambda q_1 + (1 - \lambda)q_2$, and let $P = \overline{f(p)f(q)}$. We can assume $P_1 \cap P_2 = \{p\}$, as if this were not true, then without loss of generality $P_1 \subseteq P_2$. Then $\star$ is trivial for q , as $P_1 \subseteq P \subseteq P_2$. We abuse notation and represent P_1 and P_2 as lines $[0, 1] \rightarrow \mathbb{R}^2$ via $P_i(t) = t \cdot f(q_i) + (1 - t) \cdot f(p)$. Then consider $P' : [0, 1] \rightarrow \mathbb{R}^2$ via $P'(t) = \lambda P_1(t) + (1 - \lambda)P_2(t)$. This map is a line with endpoints p and q . Thus, $P' = P$, since there is a unique line between endpoints in $\mathbb{R}^2$.

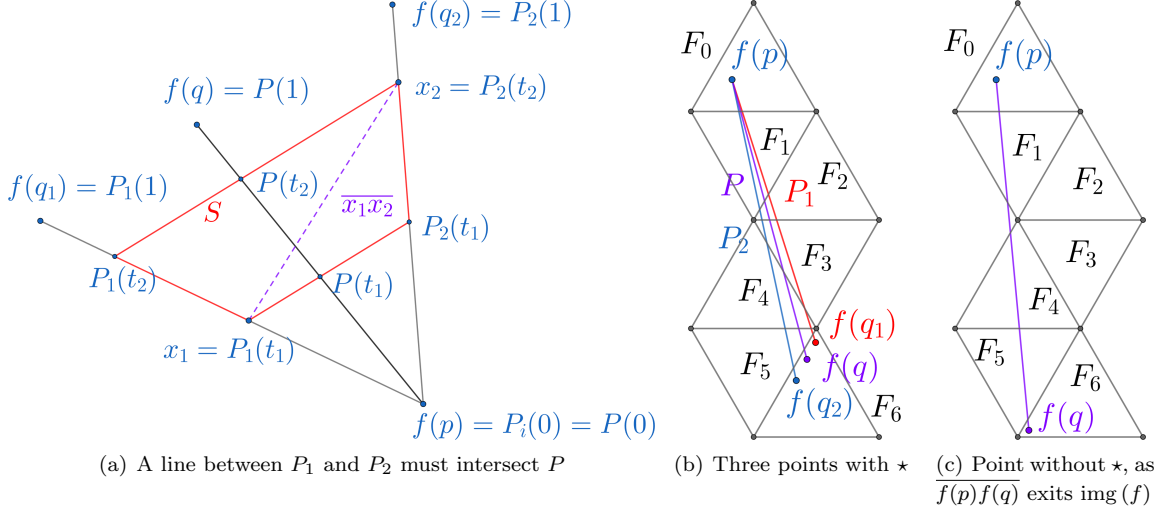


Figure 3: Visualizations for Lemma 2.5. Face paths in (b) and (c) are obtained from an icosahedron.

We will show for any $x_1 \in \text{img}(P_1)$ and $x_2 \in \text{img}(P_2)$, $\overline{x_1 x_2}$ intersects P . Let $x_1 = P_1(t_1)$ and $x_2 = P_2(t_2)$ for $t_1, t_2 \in [0, 1]$. This is trivial for $t_1 = t_2$ as we can set $t = t_1 = t_2$ and obtain $P(t) = \lambda P_1(t_1) + (1 - \lambda)P_2(t_2)$. Additionally, if $t_i = 0$ for either i , then $x_i = P_i(0) = f(p) = P(0)$. Then without loss of generality we will assume $0 < t_1 < t_2$. Since $0 < t_1, t_2$, $P_1(t_1) \neq P_2(t_1)$ and $P_1(t_2) \neq P_2(t_2)$, so $P_1(t_1), P_1(t_2), P_2(t_2), P_2(t_1)$ are four distinct points, and form a cycle S in $\mathbb{R}^2$. Note that S is formed from the triangle $f(p), P_1(t_2), P_2(t_2)$ by adding the edge $\overline{P_1(t_1)P_2(t_1)}$ and removing the segments $\overline{P_1(0)P_1(t_1)}$ and $\overline{P_1(0)P_2(t_1)}$. Thus, S is a polygon, and since any internal angle of S must be at most π , S is convex. We consider the line $\overline{x_1 x_2}$, and the line $\overline{P(t_1)P(t_2)}$. By convexity, both of these lines are contained in S . The order of their endpoints on the boundary of S is $x_1, P(t_1), x_2, P(t_2)$, so these lines must intersect. We visualize this in Figure 3(a).

We will inspect $P \cap f(F_0)$. This path certainly begins at $f(p)$. From assumptions, we know that $P_1 \cap f(F_0)$ and $P_2 \cap f(F_0)$ have endpoints on edge $f(F_0 \cap F_1)$. Thus, a convex combination of these endpoints must be on P , implying P intersects that edge. This intersection must be the first time P intersects an edge since P begins within $f(F_0)$. We will proceed with induction on $i \in \{1, \dots, n - 1\}$. We use the hypothesis that $P \cap f(F_{i-1})$ has an endpoint on $f(F_{i-1} \cap F_i)$. Since both P_1 and P_2 have endpoints on edge $f(F_i \cap F_{i+1})$, we may repeat the argument from the F_0 case to find that $P \cap f(F_i)$ must have an endpoint on $f(F_i \cap F_{i+1})$. Finally, for $P \cap f(F_n)$, we have from induction that the startpoint is on edge $f(F_{n-1} \cap F_n)$. Since P ends at point $f(q)$, we have from Lemma 2.3 that $P \cap f(F_n)$ must be a line from the startpoint to $f(q)$. This shows properties (b) and (c). Property (a) follows from noticing that during the proof for (b) and (c), we have accounted for all of line P as a sequence of line segments whose union is a line from $f(p)$ to $f(q)$. ■

We visualize an example in Figure 3(b), for a path unfolding from an icosahedron.

Definition 2.8. For metric space X , let $P \subseteq X$ be a finite collection of points. For each $p \in P$, the associated **Voronoi cell** is the set of all points in X closest to p , that is, $\left\{x \in X : d(x, p) = \min_{p' \in P} d(x, p')\right\}$. The collection of all Voronoi cells is called the **Voronoi diagram**, and covers X [13].

Lemma 2.6. For some metric space X , let $P \subseteq P' \subseteq X$ be collections of points with Voronoi diagrams V and V' respectively. Let $p \in P$ have Voronoi cell C_p in V , and cell C'_p in V' . Then $C'_p \subseteq C_p$.

Proof: By definition, if a point $x \in C'_p$, it is closest to p among all points in P' . Thus, x must be the closest to p among all points in P , since $P \subseteq P'$. Then $x \in C_p$. ■

Definition 2.9. Let X be a metric space, and take some $p \in X$. The **cut locus** of p is the set of all $q \in X$ such that there are multiple distinct geodesics from p to q . The **total cut locus** is $\{(p, q) \in X \times X : q \text{ is on cut locus of } p\}$.

Lemma 2.7. For a convex polyhedron, the cut locus on face F^* of point p (on face F_0) has the form of the boundaries of a Voronoi diagram in $\mathbb{R}^2$, and the points that generate this Voronoi diagram are copies of p arising from path unfoldings of face paths from F_0 to F^* . Additionally, for each copy p' of p with associated path unfolding f , and every point q of its Voronoi cell in $f(F^*)$, $\overline{p'q}$ is the image of a geodesic between p and $(f|_{F^*})^{-1}(q)$, and q has property $\star$ with respect to p' .

The first part of this is shown by Agarwal in [1], and properties (b) and (c) of $\star$ arise from the face path of each p' being constructed so that geodesics from p' visit the faces in assigned order. Given that $\overline{p'q}$ is the image of a geodesic, it is trivial to show (a), as $\overline{p'q}$ is certainly contained within the image of its associated path unfolding. ■

2.3.2 Cut Locus Algorithm

The total cut locus of a space is relevant to its geodesic complexity, as it represents areas where GMPRs must make a choice, and thus could potentially be discontinuous. Because of this, simplex embeddings we construct for Theorem 2.1 will be made with the total cut locus of each space in mind. Due to the significance of the cut locus, we give an algorithm³ to calculate the cut locus of a point on a polyhedron surface, and will proceed to show its correctness.

Filter Algorithm: We will show correctness of Algorithm 1, designed to filter face paths $F_0, \dots, F_n$ based on whether certain points in face F_n have property $\star$. To show termination, it is enough to show that the loop on line 2 is over a finite set. The set $f(F_n)$ is a polygon, and cells of a Voronoi diagram of a finite point set in $\mathbb{R}^2$ have boundaries composed of line segments, rays, and lines. Thus, $C \cap f(F_n)$ must also be a polygon, so V is well defined and finite. If $C \cap f(F_n)$ is empty, we return true, which is consistent with our desired output. Otherwise, since both C and $f(F_n)$ are convex (cells of a Voronoi diagram in $\mathbb{R}^2$ are convex), $C \cap f(F_n)$ is a convex polygon. Thus, by Lemma 2.5, all points in $C \cap f(F_n)$ have property $\star$ if and only if all vertices of $C \cap f(F_n)$ have property $\star$. Then our algorithm returns the correct output, since the loop on line 2 tests if q has property $\star$.

In practice, we compute line 1 by considering all vertices of C and $f(F_n)$, as well as intersections of their boundaries. We keep vertices that lie in both regions. We check line 3 by splitting $\overline{p^*q}$ into segments

³Our Python implementation is available at [14].

Algorithm 1 Filter Algorithm (Pseudocode)

Input:

$F_0, \dots, F_n$: Face path in a convex polyhedron
 $f: \bigcup F_i \rightarrow \mathbb{R}^2$: Path unfolding of $F_0, \dots, F_n$
 $p^* \in f(F_0)$: Point to inspect
 $C \subseteq \mathbb{R}^2$: Voronoi cell with $p^* \in C$

Output:

Whether $\forall q \in C \cap f(F_n)$, q has property $\star$ with respect to p^*

```
1:  $V \leftarrow$  vertices of polygon  $C \cap f(F_n)$ 
2: loop for  $q \in V$ 
3:   if  $\overline{p^*q} \not\subseteq \text{img}(f)$  then return False ▷ Test property (a)
4:    $s \leftarrow p^*$ 
5:   loop for  $i \in (0, \dots, n-1)$  ▷ Test property (c), part of (b)
6:     if  $\overline{p^*q} \cap f(F_i)$  is not a path from  $s$  to  $f(F_i \cap F_{i+1})$  then return False
7:      $s \leftarrow$  endpoint of  $\overline{p^*q} \cap f(F_i)$ 
8:   if  $\overline{p^*q} \cap f(F_n)$  is not a path from  $s$  to  $f(q)$  then return False ▷ Finish test of property (b)
9: return True
```

based on intersections with face boundaries, then verifying each segment is contained in a face. For every equivalence or incidence check in lines 6 and 8, we use a tolerance to mitigate rounding errors.

Checking if a line segment is within a face can be done in linear time wrt. the face's edge count, and checking if a point is on a line can be done in constant time. Repeated applications of these allow lines 3, 6, and 8 to be implemented in polynomial time wrt. the number of edges in all of $F_0, \dots, F_n$. Thus, considering the loops on lines 2 and 5, the overall complexity of Algorithm 1 is polynomial with respect to the number of edges in all of $F_0, \dots, F_n, C$.

Cut Locus Algorithm: We will show correctness of Algorithm 2, designed to obtain the cut locus on face F_* of a point $p \in F_0$. The algorithm terminates since the size of finite set S decreases each loop. From Lemma 2.7, the cut locus of point p must have the structure of a Voronoi diagram in $\mathbb{R}^2$. The diagram is generated by images of p from path unfoldings of face paths from F_0 to F_* . Let P^* be these images, and let V^* be the resulting Voronoi diagram. Before the loop on line 4, let $P = \{f(p) : (\mathbb{P}, f) \in S\}$, and observe $P^* \subseteq P$, since P contains all possible images of p from path unfoldings. For $x \in P^*$, we will denote C_x^* as the Voronoi cell of x in V^* , restricted to face F_* . Similarly, for $x \in P$, we will denote C_x as the Voronoi cell of x in V , restricted to face F_* .

Lemma 2.8. *For $x \in P$ such that C_x is nonempty, let $\mathbb{P}$ be the face path associated with x , and let f be the path unfolding. We claim $x \in P^*$ if and only if for all $q \in C_x$, q has property $\star$ with respect to $\mathbb{P}$.*

Proof: One direction is simple, as for $x \in P^*$ and $q \in C_x$, Lemma 2.6 shows $q \in C_x^*$, and thus by Lemma 2.7, q has property $\star$. For the other direction, let $x \in P \setminus P^*$ and $q \in C_x$, and assume for the sake of contradiction q has property $\star$. Since the Voronoi diagram V^* covers F_* , we can choose some $x^* \in P^*$ such that $q \in C_x^*$. Now x^* must also contribute to V , as $P^* \subseteq P$. Thus, by definition of Voronoi diagram, the length of $\overline{xq}$ is at most the length of $\overline{x^*q}$. Since f and its inverse preserve the lengths of paths, the preimage of $\overline{xq}$ is at most the length of the preimage of $\overline{x^*q}$. Then the preimage of $\overline{xq}$ is a geodesic, and x should be included in P^* , which contradicts our assumption. ■

Algorithm 2 Cut Locus Algorithm (Pseudocode)

Input:

$F_0 \subseteq X$: Source face of convex polyhedron X
 $p \in F_0$: Point to inspect
 $F_* \subseteq X$: Sink face of X
 $i_*: \hookrightarrow \mathbb{R}^2$: Embedding of F_*

Output:

Cut locus of p on face F_*

Finite set S with elements $(\mathbb{P}, f)$ such that:

- $\mathbb{P}$ is a face path from F_0 to F_*
- $f: \bigcup_{F \in \mathbb{P}} F \rightarrow \mathbb{R}^2$ is the path unfolding of $\mathbb{P}$ wrt. i_*
- The cut locus of p on F_* is the Voronoi diagram of $\{f(p) : (\mathbb{P}, f) \in S\}$

```
1:  $S \leftarrow \{(\mathbb{P}, f) : \mathbb{P} \text{ is a face path from } F_0 \text{ to } F_*, f \text{ is the path unfolding of } \mathbb{P} \text{ wrt. } i_*\}$ 
2:  $V \leftarrow$  Voronoi cell diagram of  $\{f(p) : (\mathbb{P}, f) \in S\}$ 
3: For each  $(\mathbb{P}, f) \in S$ ,  $C_{f(p)} \leftarrow$  Voronoi cell of  $f(p)$  in  $V$ 
4: loop until  $\forall (\mathbb{P}, f) \in S$ , (Algorithm 1)( $\mathbb{P}, f, f(p), C_{f(p)}$ ) is true
5:    $(\mathbb{P}', f') \leftarrow$  element of  $S$  such that (Algorithm 1)( $\mathbb{P}', f', f'(p), C_{f'(p)}$ ) is false
6:    $S \leftarrow S \setminus \{(\mathbb{P}', f')\}$ 
7:    $V \leftarrow$  Voronoi cell diagram of  $\{f(p) : (\mathbb{P}, f) \in S\}$ 
8:   For each  $(\mathbb{P}, f) \in S$ ,  $C_{f(p)} \leftarrow$  Voronoi cell of  $f(p)$  in  $V$ 
9: return  $(i_*^{-1}(V), S)$ 
```

The condition $\forall q \in C_x$, q has property $\star$ is exactly the condition checked by Algorithm 1. Thus, with $P = \{f(p) : (\mathbb{P}, f) \in S\}$, if the first iteration of the loop (line 4) does not run, every $x \in P$ either is in P^* or has C_x empty. This implies the cut locus restricted to face F_* is identical to V^* , and we return the correct result. On the other hand, assume $\exists x \in P$ with associated face path $\mathbb{P}$ and path unfolding f such that (Algorithm 1)($\mathbb{P}, f, x, C_x$) is false. Then by Lemma 2.8, $x \notin P^*$, so when we remove this element in line 6, we maintain that $P^* \subseteq P$.

Then it is always true that $P^* \subseteq P$, and we exit the loop only if our output is correct. Since the algorithm must terminate, it will always return the correct output. Before the final line, we may remove all $(\mathbb{P}, f) \in S$ such that $C_{f(p)} \cap V$ is empty, as this will simplify S without changing the validity of the algorithm.

Line 1 can be done with an exhaustive search over the finite graph described in Definition 2.5. For lines 2 and 7, there are many existing algorithms [3, 8, 9, 13, 16, 17] to find Voronoi diagrams in $\mathbb{R}^2$.

The output of the algorithm is can be visualized as in Figure 7(b, c), where we have all points and path unfoldings contributing to the cut locus on a particular face. We may use this algorithm on all faces of the polyhedron to produce a result as in Figure 7(a). We may also plot relevant path unfoldings starting from the source face F_0 , and transform the cut loci on each face appropriately, as in Figure 7(d). After removing unnecessary edges, this becomes a **Voronoi star unfolding** of the polyhedron, described in [1].

Since line 1 enumerates all possible face paths between two faces of X , it has exponential complexity wrt. the number of faces in X . Thus, Algorithm 2 must also be at least exponential complexity. This will limit its usability for polyhedra with many faces. With our implementation, we find that it is feasible to calculate cut loci on the icosahedron and dodecahedron within a few seconds.

3 Examples

3.1 n -Torus

We will use Theorem 2.1 to show the geodesic complexity of an n -torus equipped with the flat metric is at least n (any geodesic motion planner needs at least $n + 1$ sets). This result was initially shown by Recio-Mitter [15], and we use this as the first example since the collections of simplex embeddings we use are instructive and easy to visualize.

We represent the n -torus $X := (S^1)^n$ with elements in $(\mathbb{R}/\mathbb{Z})^n$, identifying $(\dots, x, \dots) \sim (\dots, x + m, \dots)$ for $m \in \mathbb{Z}$ in each dimension. All geodesics between $p, q \in X$ are diagonal segments that project to a geodesic in each dimension. We say a path from p to q goes ‘down’ from p in dimension k if the image in this dimension contains interval $(p_k - \delta, p_k]$ for some $\delta > 0$. We say the path goes ‘up’ from p otherwise.

Pick any $p \in X$, let $q := p + (0.5, \dots, 0.5)$ be the antipodal point, and take some $0 < \varepsilon \ll 0.5$. For each $i \in \{0, \dots, n\}$, we will construct $\mathcal{F}'_i$ as embeddings to all axis-oriented ε -boxes of dimension i with q as a vertex. Formally, we build each $\mathcal{F}'_i$ as follows: For some set $D \subseteq [n]$ such that $|D| = i$, and ‘direction’ assignments $M: D \rightarrow \{0, 1\}$, we will construct an i -dimensional axis-oriented box with sides $B_{D,M,k} \subseteq S^1$ for $k \in [n]$. For $k \notin D$, $B_{D,M,k} = \{q_k\}$. For $k \in D$, $B_{D,M,k} = [q_k, q_k + \varepsilon]$ if $M(k) = 1$, and $[q_k - \varepsilon, q_k]$ otherwise. We construct $\mathcal{F}'_i$ by choosing an embedding for each possible $B_{D,M} := B_{D,M,1} \times \dots \times B_{D,M,n}$. Whenever we choose an embedding from Δ_i to an i -dimensional box $B_{D,M}$, we map i of the simplex faces as the i box faces that contain q . This ensures restrictions to these faces are represented in $\mathcal{F}'_{i-1}$.

Finally, for each $i \in \{0, \dots, n\}$, we construct $\mathcal{F}_i := \{t \mapsto (p, F(t)) : F \in \mathcal{F}'_i\}$. We display the images of $F \in \mathcal{F}'_i$ for the 2-torus in Figure 4(a). We will verify that the hypotheses of Theorem 2.1 hold.

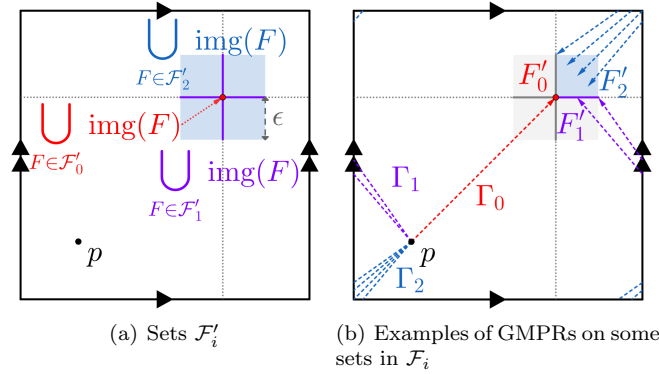


Figure 4: Visualization of sets used in the 2-torus

(a) By construction, each $\mathcal{F}_i$ contains embeddings of i -dimensional simplices.

(b) Take some $F_i \in \mathcal{F}_i$. Without loss of generality $\text{img}(F_i) = \{p\} \times \left(\prod_{k=1}^i [q_k, q_k + \varepsilon] \times \prod_{k=i+1}^n \{q_k\} \right)$, as

the other cases are symmetric. We will characterize all GMPRs on $\text{img}(F_i)$. Since geodesics in X are geodesics in each dimension, the GMPR must go ‘down’ from p in dimensions $1, \dots, i$, and can choose a direction for the others. This creates 2^{n-i} possible GMPRs, distinguished by these choices.

Continuity follows from continuity of projections to each dimension.

Let $(p, q') \in \text{relint}^*(F_i)$, and take geodesic G between them. We have $q' = ((q_k + \delta_k)_{k \in [i]}, (q_k)_{k > i})$ for $0 < \delta_k < \varepsilon$, so G must go ‘down’ from p in dimensions $1, \dots, i$. Thus, there exists a GMPR on $\text{img}(F_i)$ extending G .

- (c) Take the same $F_i \in \mathcal{F}_i$, and take a GMPR Γ_i on $\text{img}(F_i)$ that without loss of generality goes ‘up’ from p in dimension $i+1$. We will show that $F_{i+1} \in \mathcal{F}_{i+1}$ where $\text{img}(F_{i+1}) = \{p\} \times \left(\prod_{k=1}^{i+1} [q_k, q_k + \varepsilon] \times \prod_{k=i+2}^n \{q_k\} \right)$

satisfies our requirements.

- (i) $F_{i+1} \in \mathcal{F}_{i+1}$ and has F_i as a face by construction.
- (ii) Pick some geodesic G from Γ_i . We will show there is an open set separating G from any geodesic in $\pi_{GX}^{-1}(\text{relint}^*(F_{i+1}))$, which all go ‘down’ from p in dimension $i+1$. By our choice of q and ε , the image of any of these geodesics in dimension $i+1$ intersects $(q_{i+1}, q_{i+1} + \varepsilon)$ and does not intersect $(q_{i+1} - \varepsilon, q_{i+1})$. In contrast, the image of G in dimension $i+1$ intersects only $(q_{i+1} - \varepsilon, q_{i+1})$. Let $t_0 \in [0, 1]$ such that $G(t_0)_{i+1} \in (q_{i+1} - \varepsilon, q_{i+1})$. We construct an open set $U \subseteq GX$ via $U = \{G' \in GX : G'(t_0)_{i+1} \in (q_{i+1} - \varepsilon, q_{i+1})\}$. From the discussion above, $G \in U$, while no geodesics in $\pi_{GX}^{-1}(\text{relint}^*(F_{i+1}))$ are in U .
- (iii) There are $2^{n-(i+1)}$ GMPRs on $\text{img}(F_{i+1})$, distinguished by choice of direction in dimensions $i+2, \dots, n$.

These show hypothesis (c). We visualize an example on the 2-torus in Figure 4(b), where for step $0 \leq i < 2$, we arbitrarily choose Γ_i , one of 2^{2-i} possible GMPRs on the image of $F_i \in \mathcal{F}_i$, and choose an F_{i+1} that ‘avoids’ Γ_i . We display each Γ_i , as well as Γ_2 , the only GMPR on $\text{img}(F_2)$. With our construction, $\text{img}(F_i) = \{p\} \times F'_i$ for some axis-oriented box F'_i , so we also display each F'_i .

Thus, $GC((S^1)^n) \geq n$, so any geodesic motion planner needs at least $n+1$ sets. We have $GC((S^1)^n) = n$ from the explicit geodesic motion planner by Cohen and Pruidze [4].

3.2 Tetrahedron

We show how the results of Davis [5] fit into this framework, and show that the geodesic complexity of a tetrahedron is at least three (needing at least four sets in any geodesic motion planner).

3.2.1 Cut Locus

First, we will calculate the cut locus of a point p on the tetrahedron X . We represent all faces as equilateral triangles with side length $2\sqrt{3}$, oriented as in Figure 7(a) with the face’s center point at $(0, 0)$. We label the faces Face 0, $\dots$, Face 3 as in Figure 7(a). When considering walk unfoldings to a particular face (as in Figure 7(b, c, d)), we use the walk unfolding that preserves the center point and orientation of that face.

Up to symmetry, a point on one of six congruent triangles on a face (displayed in Figure 5) represents any point on the tetrahedron surface. Thus, we need only calculate the cut locus on one triangle. We choose closed set τ as the bottom left region of Face 3 with vertices a , C , and M , and first consider $p \in \text{int}(\tau)$. From our implementation of Algorithm 2, we obtain the cut locus on each face (displayed in Figure 7), as well as the path unfoldings that contribute to them. We find the cut locus appears on Faces 0 and 1.

Let $R_\theta := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ be the rotation matrix, so $R_\theta p$ represents a rotation of p by θ . From Algorithm 2,

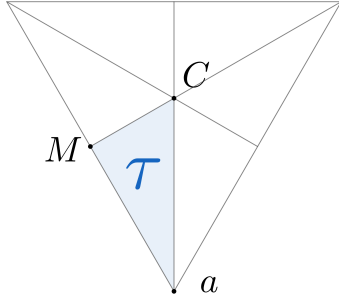


Figure 5: Representative region of the tetrahedron

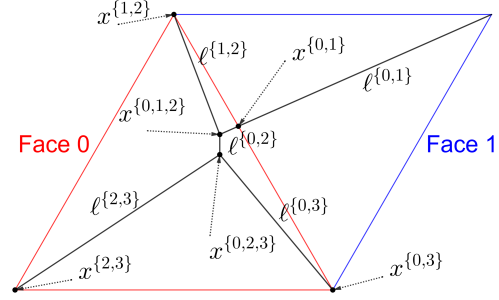


Figure 6: Labeled ℓ^I and $x^{I'}$ on cut locus of $p = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) \in \text{int}(\tau)$

we find the cut locus on Face 0 of p arises from the Voronoi diagram of four points, displayed in Figure 7(b).

We enumerate these points $(p^{(0)}, p^{(1)}, p^{(2)}, p^{(3)}) = \left(\begin{bmatrix} 2\sqrt{3} \\ 0 \end{bmatrix} + R_\pi p, \begin{bmatrix} 2\sqrt{3} \\ 4 \end{bmatrix} + p, \begin{bmatrix} -2\sqrt{3} \\ 0 \end{bmatrix} + R_\pi p, \begin{bmatrix} 0 \\ -2 \end{bmatrix} + p \right)$.

We denote $\ell^{\{i,j\}}(p)$ as the line that bisects $p^{(i)}$ and $p^{(j)}$. Note that which $\ell^{\{i,j\}}(p)$ exist on the cut locus depend on the choice of p . This is displayed in Figure 9 (a) and (c), where $\ell^{\{0,2\}}$ is replaced by $\ell^{\{1,3\}}$.

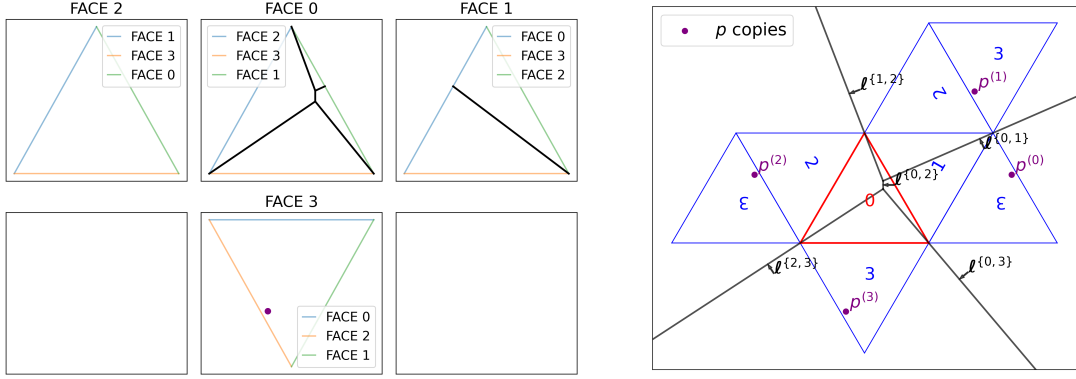
We denote $x^{\{i,j\}}(p)$ as a specified intersection of $\ell^{\{i,j\}}(p)$ with an edge of Face 0, and $x^{\{i,j,k\}}(p)$ as the intersection of $\ell^{\{i,j\}}(p)$ and $\ell^{\{j,k\}}(p)$. If two points coincide $x^I(p) = x^{I'}(p)$, we call this point $x^{I \cup I'}(p)$. We choose our notation so that $x^I(p)$ is equidistant from points $(p^{(i)})_{i \in I}$. We display ℓ^I and $x^{I'}$ for a point $p \in \text{int}(\tau)$ in Figure 6. We will calculate the locations of some of these lines and intersecting points.

- $\ell^{\{0,1\}}(p)$ has equation $t \mapsto (2\sqrt{3}, 2) + ((-2, 0) + R_{\pi/2}p)t$.
Let $x^{\{0,1\}}(p)$ be the intersection of $\ell^{\{0,1\}}(p)$ with the right edge of Face 0.
- $\ell^{\{1,2\}}(p)$ has equation $t \mapsto (0, 2) + ((-2, 2\sqrt{3}) + R_{\pi/2}p)t$.
 $\ell^{\{1,2\}}(p)$ always intersects the top vertex of the triangle. Let this point be $x^{\{1,2\}}(p) := (0, 2)$.
- $\ell^{\{2,3\}}(p)$ has equation $t \mapsto (-\sqrt{3}, -1) + ((1, \sqrt{3}) + R_{\pi/2}p)t$.
 $\ell^{\{2,3\}}(p)$ always intersects the left vertex of the triangle. Let this point be $x^{\{2,3\}}(p) := (-\sqrt{3}, -1)$.
- $\ell^{\{0,3\}}(p)$ has equation $t \mapsto (\sqrt{3}, -1) + ((1, -\sqrt{3}) + R_{\pi/2}p)t$.
 $\ell^{\{0,3\}}(p)$ always intersects the right vertex of the triangle. Let this point be $x^{\{0,3\}}(p) := (\sqrt{3}, -1)$.

Using SymPy, a symbolic calculator [12], we find the other x^I by calculating intersections of lines:

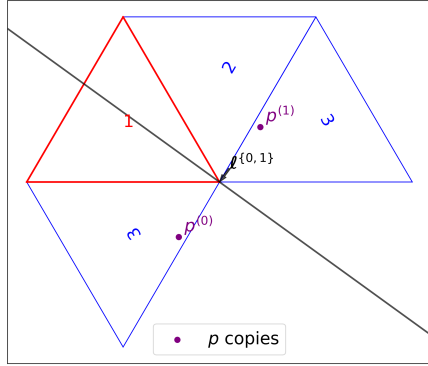
- $x^{\{0,1\}}(p) = \left(\frac{2\sqrt{3}p_1}{p_1 - \sqrt{3}p_2 - 2\sqrt{3}}, 2 \frac{2p_1 + \sqrt{3}p_2 + 2\sqrt{3}}{-p_1 + \sqrt{3}p_2 + 2\sqrt{3}} \right)$.
- $x^{\{0,1,2\}}(p) = \left(-p_1, \frac{p_1^2 + 2\sqrt{3}p_1 + 2p_2 + 4}{p_2 + 2} \right)$.
- $x^{\{0,2,3\}}(p) = \left(-p_1, \frac{p_1^2 - p_2 - 2}{p_2 - 1} \right)$.
- $x^{\{0,1,3\}}(p) = \left(\frac{\sqrt{3}p_1p_2 - 4\sqrt{3}p_1 + 3p_2^2 + 6p_2}{-3p_1 + \sqrt{3}p_2 + 2\sqrt{3}}, \frac{-\sqrt{3}p_1^2 - 3p_1p_2 + 2\sqrt{3}p_2 + 4\sqrt{3}}{-3p_1 + \sqrt{3}p_2 + 2\sqrt{3}} \right)$.
- $x^{\{1,2,3\}}(p) = \left(\frac{-\sqrt{3}p_1p_2 - 2\sqrt{3}p_1 - 3p_2^2 - 6p_2}{3p_1 - \sqrt{3}p_2 + 4\sqrt{3}}, \frac{\sqrt{3}p_1^2 + 3p_1p_2 + 12p_1 + 4\sqrt{3}p_2 + 8\sqrt{3}}{3p_1 - \sqrt{3}p_2 + 4\sqrt{3}} \right)$.

Using our calculated expressions, we will show that the cut locus structure in Figure 7(b) is the same across the interior of τ . We do this by verifying the following are true:

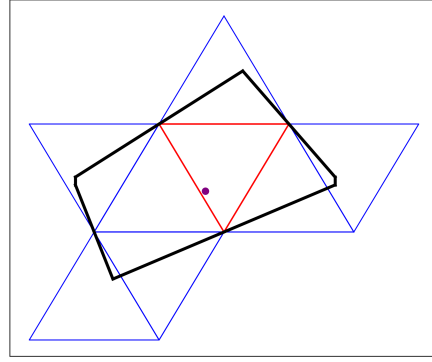


(a) Labeling of tetrahedron faces, boundaries between them, and the cut locus of p on each of them

(b) Copies of p and path unfoldings that contribute to the cut locus on Face 0



(c) Copies of p and path unfoldings that contribute to the cut locus on Face 1



(d) Voronoi star unfolding from p

Figure 7: Output of Algorithm 2 on a tetrahedron for $p = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) \in \text{int}(\tau)$ on Face 3

- Lines $\ell^{\{0,1\}}$, $\ell^{\{1,2\}}$, $\ell^{\{2,3\}}$, and $\ell^{\{0,3\}}$ exist on the cut locus and are rays in the corresponding Voronoi diagram.
- Lines $\ell^{\{0,1\}}$ and $\ell^{\{1,2\}}$ end at point $x^{\{0,1,2\}}$, and this point is distinct from other $x^{\{0,1,k\}}$ or $x^{\{1,2,k\}}$.
- Lines $\ell^{\{2,3\}}$ and $\ell^{\{0,3\}}$ end at point $x^{\{0,2,3\}}$, and this point is distinct from other $x^{\{2,3,k\}}$ or $x^{\{0,3,k\}}$.
- Points $x^{\{0,1,2\}}$ and $x^{\{0,2,3\}}$ are on the interior of Face 0.

We claim this is sufficient to fully prove the structure of the cut locus on Face 0, as we can account for the boundaries of all four cells. For example, the cell corresponding to $p^{(1)}$ must be partially bounded by $\ell^{\{0,1\}}$ and $\ell^{\{1,2\}}$. If these are both rays that begin at the same point, this must account for the entire boundary of this cell. Similar arguments account for the full boundary of cells corresponding to $p^{(0)}$, $p^{(2)}$, and $p^{(3)}$.

To show these hypotheses, we utilize the Wolfram Mathematica symbolic calculator [10], which can find regions in $\mathbb{R}^2$ where inequalities hold.

- The fact that a line $\ell^{\{i,j\}}$ exists on the cut locus may be verified by simply checking that a point $m \in \ell^{\{i,j\}}$ is closer to $p^{(i)}$ and $p^{(j)}$ than any other copy of p . Specifically, we verify for all $p \in \text{int}(\tau)$ that $d(p^{(i)}, m) < d(p^{(k)}, m)$ for all $k \neq i, j$. We pick m using our calculation of $\ell^{\{i,j\}}$ and its endpoints.
- To verify that $\ell^{\{i,j\}}$ is a ray on the cut locus, it is enough to show that it contains a ray, and has an endpoint. We pick a point $m \in \ell^{\{i,j\}}$ and a predicted direction vector $v \in \mathbb{R}^2$, where we expect $\ell^{\{i,j\}}$ to contain $m + tv$ for $t \geq 0$. As before, we verify m is on the cut locus line corresponding to $\ell^{\{i,j\}}$. From this, it is enough to show that for $k \neq j$, $\ell^{\{i,k\}}$ does not intersect $\{m + tv : t \geq 0\}$, since if a line segment on a Voronoi diagram in $\mathbb{R}^2$ ends, it must be at a vertex. Thus, we verify for all $p \in \text{int}(\tau)$ that for all $k \neq i, j$, we have $v \cdot (x^{\{i,j,k\}} - m) < 0$. To ensure we have a ray and not a line, we must verify that an endpoint $x^{\{i,j,k\}}$ of $\ell^{\{i,j\}}$ is on the cut locus, which can be done by showing for $p \in \text{int}(\tau)$, that $d(p^{(i)}, x^{\{i,j,k\}}) \leq d(p^{(k')}, x^{\{i,j,k\}})$ for all $k' \neq i, j, k$. To show that this point is distinct from any other $x^{\{i,j,k'\}}$, it is enough to show this inequality is strict.
- Finally, checking if a point $x^{\{i,j,k\}}$ is on the interior of each edge on Face 0 can be done using a point on the edge and the edge's normal vector.

We find that all hypotheses hold on the interior of τ , showing the cut locus on Face 0 is as expected. We find the cut locus on Face 1 arises from the Voronoi diagram of two points, as in Figure 7(c). We also find that the line $\ell^{\{0,1\}}$ is equivalent to the line of the same name in Figure 7(b). Thus, we can extend the same analysis to compute the entire cut locus of any $p \in \text{int}(\tau)$.

Similar calculations for each of the edges and vertices of τ show that the cut loci from Algorithm 2 match the results in the literature. We display examples of these cut loci in Figure 8, and in Appendix A, we reproduce all figures from [5, Sec. 2]. We also verify that when choosing p within aM , the cut locus on Face 0 is created from four copies of p . These copies arise from the same path unfoldings as the copies that create the cut locus within τ . Thus, as p varies from the interior of τ to aM , two vertices of the cut locus approach each other, causing line $\ell^{\{0,2\}}$ to disappear from the cut locus (see Figure 9(a) and (b)).

3.2.2 Geodesic Complexity Lower Bound

We will define sets $\mathcal{F}_i$ to use Theorem 2.1. To do this, we will embed simplices into $X \times X$ using Lemma 2.1.

First, split Face 2 into six congruent triangles, as in Figure 5. Let τ' be the triangle that shares edge aM with τ . Let $p^* \in \text{relint}^*(aM)$, and let q^* be the vertex of the corresponding cut locus (see Figure 8(d)). Let $p_3: [0, 1] \hookrightarrow X$ be an embedding so that $p_3(1) = p^*$ and $p_3([0, 1)) \subseteq \text{int}(\tau)$. Similarly, let $p_2: [0, 1] \hookrightarrow X$ so that $p_2(1) = p^*$ and $p_2([0, 1)) \subseteq \text{int}(\tau')$. This is possible, a simple example would be linear paths between the points in Figure 9(a) and (b) for p_3 , and between the points in Figure 9(c) and (b) for p_2 .

Let $\mathcal{J} := \{J \subseteq \{0, 1, 2, 3\} : |J| = 3\}$, and $\mathcal{I} := \{\{i, j\} \subseteq \{0, 1, 2, 3\} : j \equiv i + 1 \pmod{4}\}$ be collections of subsets of $\{0, 1, 2, 3\}$. For $F \in \{2, 3\}$, $I \in \mathcal{I}$, and $J \in \mathcal{J}$, with $I \subseteq J$, we will construct map $q^{F,I,J}: [0, 1] \rightarrow X$ to vary linearly between $x^I \circ p_F$ and $x^J \circ p_F$. Explicitly, $q^{F,I,J}(t) = (1-t) \cdot x^I(p_F(t)) + t \cdot x^J(p_F(t))$. Since x^I and x^J are both on line ℓ^I , $q^{F,I,J}$ will always be on line ℓ^I . Additionally, $q^{F,I,J}(1) = x^J(p_F(1)) = q^*$.

We will describe one choice of F , I , and J . We assume p_3 and p_2 are linear paths suggested by Figure 9. Let $F = 3$, $I = \{0, 1\}$, and $J = \{0, 1, 2\}$. The line p_3 varies from the location of p in Figure 9(a) to the location in Figure 9(b). The point $x^{\{0,1\}}$ is the intersection of $\ell^{\{0,1\}}$ with the edge of Face 0, and $x^{\{0,1,2\}}$ is the top vertex of the cut locus in Figure 9(a). Thus, $q^{3,\{0,1\},\{0,1,2\}}$ varies between these two points, staying on line $\ell^{\{0,1\}}$ the whole time. We display a few locations of $q^{3,\{0,1\},\{0,1,2\}}$ in Figure 10.

We will now describe sets $\mathcal{F}_i$. Recall that Lemma 2.1 takes an embedding $T_k \hookrightarrow Z$ and a map $T_k \hookrightarrow Y^{k'}$

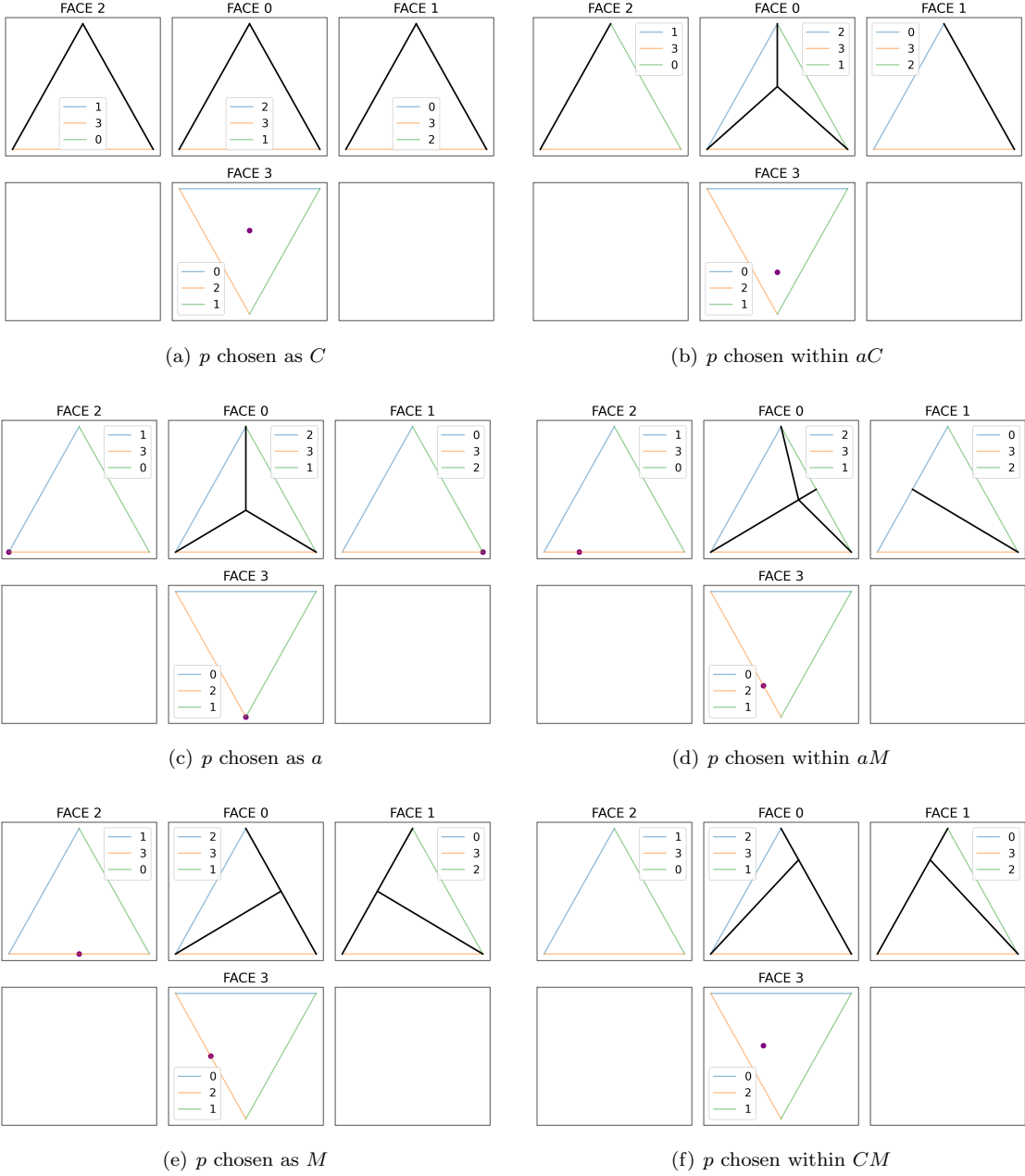
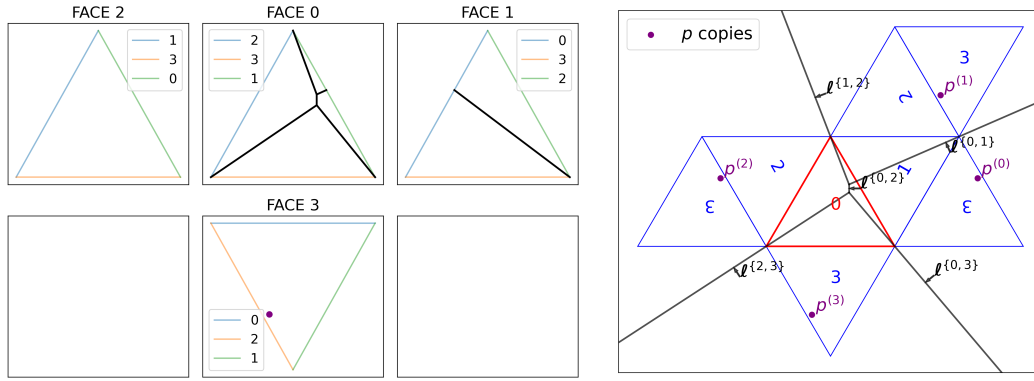
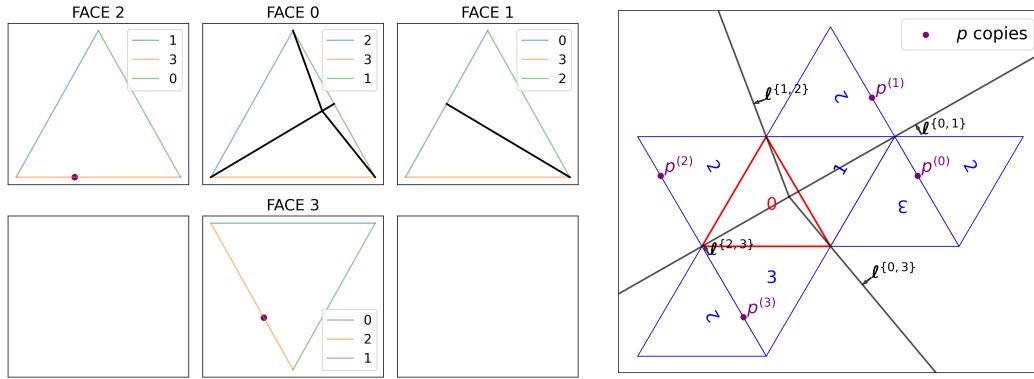


Figure 8: Cut locus of p chosen on the boundary of τ

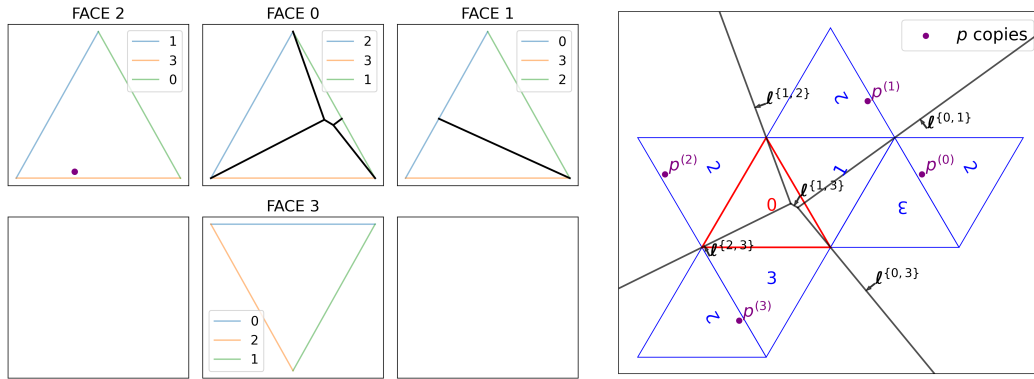
to define a simplex embedding $\Delta_{k+k'-1} \hookrightarrow Z \times Y$. In this case, we will set $k = 1$ and $Z = X$ so that $T_k = [0, 1]$, and p_3 or p_2 may be used as the first embedding. Since each face of X is a convex subset of $\mathbb{R}^2$, we let Y be Face 0, resulting in our desired embedding $\Delta_{k+k'-1} \hookrightarrow X \times X$.



(a) p chosen within τ on Face 3



(b) p chosen within line aM



(c) p chosen within τ' on Face 2

Figure 9: Cut locus on Face 0 while varying p

- $\mathcal{F}_0 = \{\Delta_0 \mapsto (p^*, q^*)\}$.
- $\mathcal{F}_1$ will be defined by four lines.
Consider p_3 , and the map $x^J \circ p_3$ for $J \in \{\{0, 1, 2\}, \{0, 2, 3\}\}$. By Lemma 2.1, this describes an embedding $f: \Delta_1 \hookrightarrow X \times X$ where $\pi_1(f(t, 1-t)) = p_3(t)$, and $\pi_2(f(t, 1-t)) = x^J(p_3(t))$. Each map follows $(p, x^J(p))$ as p approaches its endpoint on aM .
We define the other two embeddings similarly using p_2 and $x^J \circ p_2$ for $J \in \{\{0, 1, 3\}, \{1, 2, 3\}\}$.
- $\mathcal{F}_2$ will be 10 simplex embeddings, corresponding to the 5 lines around the cut locus vertices x^J considered when constructing $\mathcal{F}_1$ (each vertex is adjacent to three lines, with one line counted twice).
Consider p_3 and the map $q^{3,I,\{0,1,2\}} \boxtimes (x^{\{0,1,2\}} \circ p_3)$ for $I \in \{\{0, 1\}, \{1, 2\}\}$. Since $q^{3,I,\{0,1,2\}}(1) = x^{\{0,1,2\}}(p_3(1)) = q^*$, and these points are otherwise distinct, we have by Lemma 2.1 that this defines an embedding $f: \Delta_2 \hookrightarrow X \times X$ whose image looks like (p, q) for $p \in p_3([0, 1])$ and $q \in \ell^I(p)$.
We can define similar maps using p_3 and the maps $q^{3,I,\{0,2,3\}} \boxtimes (x^{\{0,2,3\}} \circ p_3)$ for $I \in \{\{2, 3\}, \{0, 3\}\}$. This creates four embeddings corresponding to p_3 .
We get the remaining map by considering just the vertex embeddings $x^J \circ p_3$. Explicitly, consider p_3 and the map $(x^{\{0,1,2\}} \circ p_3) \boxtimes (x^{\{0,2,3\}} \circ p_3)$. Since $x^{\{0,1,2\}}(p_3(1)) = x^{\{0,2,3\}}(p_3(1)) = q^*$, and $x^{\{0,1,2\}} \circ p_3 \neq x^{\{0,2,3\}} \circ p_3$ otherwise, this defines an embedding $\Delta_2 \hookrightarrow X \times X$ whose image looks like (p, q) for $p \in p_3([0, 1])$ and $q \in \ell^{(0,2)}(p)$.
We create 5 embeddings corresponding to p_2 in a similar way to get a total of 10.
- $\mathcal{F}_3$ will be a total of 12 simplex embeddings, corresponding to the three Voronoi cells around each of the four cut locus vertices considered when constructing $\mathcal{F}_1$.
Consider p_3 and the map $(x^{\{0,1,2\}} \circ p_3) \boxtimes (x^{\{0,2,3\}} \circ p_3) \boxtimes q^{3,\{0,1\},\{0,1,2\}}$. Note that $x^{\{0,1,2\}}(p_3(1)) = x^{\{0,2,3\}}(p_3(1)) = q^{3,\{0,1\},\{0,1,2\}}(1) = q^*$, and these maps are otherwise not collinear. By Lemma 2.1, this defines an embedding $f: \Delta_3 \hookrightarrow X \times X$ where for each $\mathbf{d} \in \Delta_3$, $f(\mathbf{d}) = (p, q)$ for $p \in p_3([0, 1])$ and q in the convex hull of $x^{\{0,1,2\}}(p)$, $x^{\{0,2,3\}}(p)$, and $q^{3,\{0,1\},\{0,1,2\}}(\mathbf{d}_1)$.
We will create two similar embeddings by considering p_3 and the maps $(x^{\{0,1,2\}} \circ p_3) \boxtimes q^{3,\{0,1\},\{0,1,2\}} \boxtimes q^{3,\{1,2\},\{0,1,2\}}$, and $(x^{\{0,1,2\}} \circ p_3) \boxtimes q^{3,\{1,2\},\{0,1,2\}} \boxtimes (x^{\{0,2,3\}} \circ p_3)$.
These are the three embeddings corresponding to cut locus vertex $x^{\{0,1,2\}} \circ p_3$. We will obtain the embeddings corresponding to the vertices $(x^{\{0,2,3\}} \circ p_3)$, $(x^{\{0,1,3\}} \circ p_2)$, and $(x^{\{1,2,3\}} \circ p_2)$ in a similar way. This accounts for the other nine embeddings.

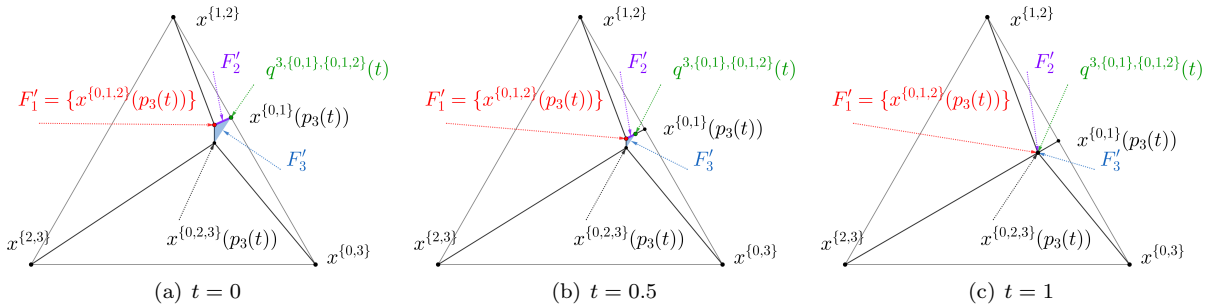


Figure 10: Visualization of selected $F_i \in \mathcal{F}_i$ as p varies towards p^*

We will verify the properties of Theorem 2.1 for a few embeddings and geodesic choices, and assert that is true for all of them by symmetry and by construction.

First, each $\mathcal{F}_i$ contains the embeddings of $\Delta_i \hookrightarrow X \times X$ by construction, so property (a) always holds.

- Consider $F_0 \in \mathcal{F}_0$. Property (b) is trivial since $\text{img}(F_0) = \{(p^*, q^*)\}$ is a single point. We will consider a GMPR Γ_0 on $\text{img}(F_0)$. We have four options, determined by which copy of p^* ($p^{(0)}$, $p^{(1)}$, $p^{(2)}$, or $p^{(3)}$) we connect q^* to. Assume our choice connects q^* to $p^{(3)}$. Then our choice for $F_1 \in \mathcal{F}_1$ will be the embedding constructed from maps p_3 and $x^{\{0,1,2\}} \circ p_3$ (e.g. (p, q) where p moves linearly from Figure 9(a) to Figure 9(b), and q is the top vertex of the resulting cut locus). (i) is satisfied by construction. (ii) is satisfied since for any element $(p, q) \in \text{relint}^*(F_1)$, q is a point on the cut locus of p with geodesics only to $p^{(0)}$, $p^{(1)}$, and $p^{(2)}$. (iii) is satisfied since there are three GMPRs on $\text{img}(F_1)$, each distinguished by which of $p^{(0)}$, $p^{(1)}$, or $p^{(2)}$ they choose to connect to. Thus, property (c) is satisfied.
- Consider $F_1 \in \mathcal{F}_1$. Property (b) is satisfied since any geodesic for $(p, q) \in \text{relint}^*(F_1)$, defined by connecting q to $p^{(0)}$, $p^{(1)}$, or $p^{(2)}$, extends to a GMPR on $\text{img}(F_1)$ distinguished by the same choice. We will consider one of the three GMPRs Γ_1 . We will choose Γ_1 to connect to $p^{(2)}$. Then our choice for $F_2 \in \mathcal{F}_2$ will be the embedding constructed from maps p_3 and $q^{3,\{0,1\},\{0,1,2\}} \boxtimes (x^{\{0,1,2\}} \circ p_3)$. This embedding follows a segment of $\ell^{\{0,1\}}(p)$ for each choice of p . (i) is satisfied by construction. (ii) is satisfied since for any element $(p, q) \in \text{relint}^*(F_2)$, q is a point on the cut locus of p with geodesics only to $p^{(0)}$ and $p^{(1)}$. (iii) is satisfied since there are two GMPRs on $\text{img}(F_2)$, each distinguished by which of $p^{(0)}$ or $p^{(1)}$ they choose to connect to. Thus, property (c) is satisfied.
- Consider $F_2 \in \mathcal{F}_2$. Property (b) is satisfied since any geodesic for $(p, q) \in \text{relint}^*(F_2)$, defined by connecting q to $p^{(0)}$ or $p^{(1)}$, extends to a GMPR distinguished by the same choice. We will consider one of the two GMPRs Γ_2 . We will choose Γ_2 to connect to $p^{(1)}$. Then our choice for $F_3 \in \mathcal{F}_3$ will be the embedding constructed from maps p_3 and $(x^{\{0,1,2\}} \circ p_3) \boxtimes (x^{\{0,2,3\}} \circ p_3) \boxtimes q^{3,\{0,1\},\{0,1,2\}}$. This embedding follows a face bounded by $\ell^{\{0,1\}}(p)$ and $\ell^{\{0,2\}}(p)$ for each choice of p . (i) is satisfied by construction. (ii) is satisfied since for any element $(p, q) \in \text{relint}^*(F_3)$, q is a point on the cut locus of p with a geodesic only to $p^{(0)}$. (iii) is satisfied since there is only one GMPR on $\text{img}(F_2)$, as the only choice that extends to the interior is to go to $p^{(0)}$. Thus, property (c) is satisfied.
- Consider $F_3 \in \mathcal{F}_3$. Property (b) is satisfied since any geodesic for $(p, q) \in \text{relint}^*(F_3)$ is defined by connecting q to $p^{(0)}$, and extends to the only GMPR over $\text{img}(F_3)$. We do not need to check (c) as 3 is our maximal dimension.

Then by Theorem 2.1, $GC(X) \geq 3$ (needing at least four sets in any geodesic motion planner).

We visualize this proof as follows. For $i > 0$, each F_i is a map $\Delta_i \rightarrow X \times X$, and is generated using Lemma 2.1 with $k = 1$. Recall that in this case, $T_1 = [0, 1]$ and for $t \in T_1$, $D_{t,i} = \{x \in \Delta_i : x_1 = t\}$. From construction, we have that the projection on the second factor $\pi_2(F_i(D_{t,i}))$ is the convex hull of i points. Let this hull be F'_i . In Figure 10, we plot the F'_i corresponding to the F_i we inspected for a few values of t . We also visualize all of $\mathcal{F}_i$ in a similar way in Appendix A.

3.3 Octahedron

We will prove Theorem 1.1, that the geodesic complexity of an octahedron is exactly four (needing five sets in an efficient geodesic motion planner).

3.3.1 Cut Locus

We will calculate the cut locus of a point p on the octahedron X . Similar to the tetrahedron, we represent all faces as equilateral triangles centered at $(0,0)$ with side length $2\sqrt{3}$, oriented and labeled as shown in Figure 12(a). We choose p to be on Face 6, and consider the corresponding cut locus.

We claim that the nineteen regions of Face 6 displayed in Figure 11(a) each have isomorphic cut loci, and the lines of these cut loci vary continuously with the choice of p . These regions consist of four points, nine line segments, and six triangles. The fact that the cut loci vary continuously with the choice of p follows from the fact that the cut loci are boundaries from a Voronoi diagram, and the points that create this diagram vary continuously. Thus, we will proceed to find the structure of the cut locus on these regions.

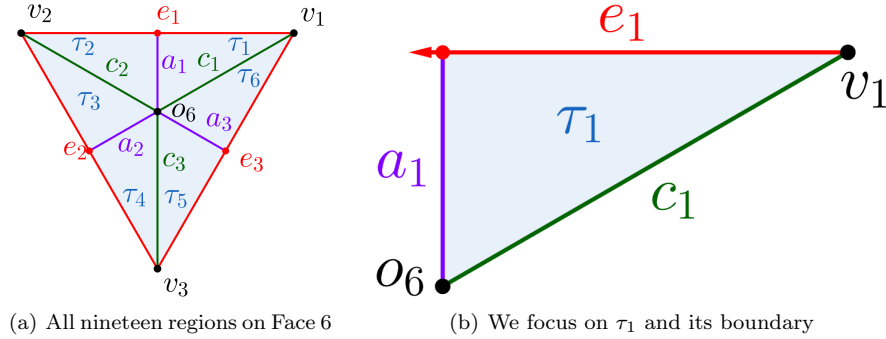


Figure 11: Partition of Face 6 into regions with isomorphic cut loci

Up to symmetry, a point on one of six congruent triangles on a face (displayed in Figure 11(a)) represents any point on the octahedron surface. Thus, we need only consider one of the τ_i , along with its boundary. We choose τ_1 , whose boundary contains a_1 , c_1 , v_1 , o_6 , and part of edge e_1 . Note that Figure 11(a) is a partition, so each a_i, c_i, e_i do not contain their endpoints, and each τ_i does not contain its boundary. We will first inspect the sets on the interior of Face 6, τ_1 , c_1 , a_1 , and o_6 . From Algorithm 2, we find that for p chosen in these regions, the cut locus appears on Faces 0, 1, 3, and 4.

On this region, the cut locus on Face 0 of p arises from the Voronoi diagram of six points, as displayed in Figure 12(b). We enumerate these $(p^{(0)}, \dots, p^{(5)}) =$

$\left(\begin{bmatrix} 3\sqrt{3} \\ 1 \end{bmatrix} + p, \begin{bmatrix} 2\sqrt{3} \\ 4 \end{bmatrix} + R_{2\pi/3}p, \begin{bmatrix} -2\sqrt{3} \\ 4 \end{bmatrix} + R_{-2\pi/3}p, \begin{bmatrix} -3\sqrt{3} \\ 1 \end{bmatrix} + p, \begin{bmatrix} -\sqrt{3} \\ -5 \end{bmatrix} + R_{2\pi/3}p, \begin{bmatrix} \sqrt{3} \\ -5 \end{bmatrix} + R_{-2\pi/3}p \right)$. Analogous to the tetrahedron, we define $\ell^{\{i,j\}}(p)$ as the line of the cut locus that bisects $p^{(i)}$ and $p^{(j)}$, $x^{\{i,j\}}(p)$ as the intersection of selected $\ell^{\{i,j\}}(p)$ with an edge of Face 0, and $x^{\{i,j,k\}}(p)$ as the intersection of $\ell^{\{i,j\}}(p)$ and $\ell^{\{j,k\}}(p)$. Using a symbolic calculator, we explicitly find their equations in Appendix B.

To find the structure of the cut loci, and that they are isomorphic within their respective regions, we use the Wolfram Mathematica symbolic calculator as we did in Section 3.2.1. For p chosen within each region, this allows us to verify which $\ell^{\{i,j\}}$ occur on the cut locus as well as the identity and location of each line's endpoints. To find the cut locus structure on Face 0, it is enough to prove the boundary of each of the six Voronoi cells in this way. We first inspect p chosen on the interior of Face 6, since we find that for these regions, the cut loci come from copies of p that arise from the same path unfoldings.

Using Algorithm 2, we find that when p is on the interior of Face 6, the cut loci on Faces 1, 3, and 4 are

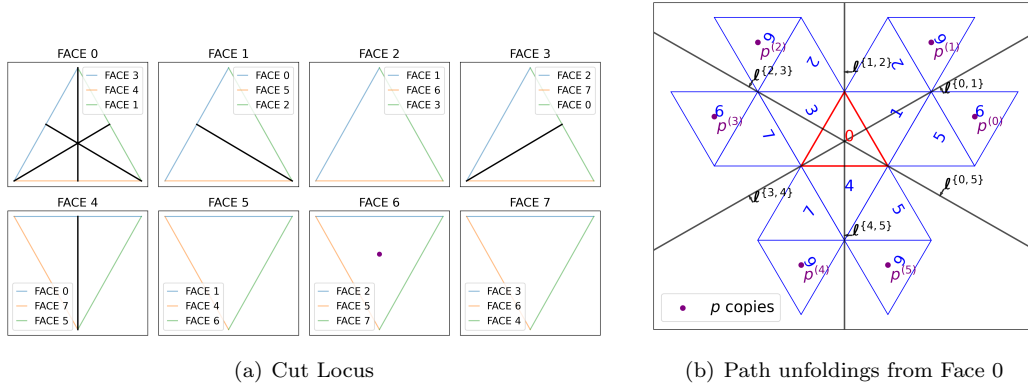


Figure 12: Cut locus of an octahedron with respect to $p = o_6$

as suggested in Figure 12(b). They each consist of a line arising from two copies of p , and are equivalent to extensions of the lines $\ell^{\{0,1\}}$, $\ell^{\{2,3\}}$, and $\ell^{\{4,5\}}$ from the cut locus on Face 0. We also observe from symmetries of the copies of p that lines $\ell^{\{0,1\}}$, $\ell^{\{2,3\}}$, and $\ell^{\{4,5\}}$ always begin at the indicated vertices of Faces 1, 3, and 4 respectively. Similarly, $\ell^{\{1,2\}}$, $\ell^{\{3,4\}}$, and $\ell^{\{0,5\}}$ always begin at the vertices of Face 0.

Point o_6 : First, we consider point o_6 , the center of Face 6. We observe that the structure of the cut locus is a star, with 6 lines joined at a point. We display this in Figure 12.

Line a_1 : We find that the structure of the cut locus of $p \in a_1$ is isomorphic to Figure 13, with one vertex $x^{\{1,2,4,5\}}$ incident to four line segments and two vertices $x^{\{0,1,5\}}$ and $x^{\{2,3,4\}}$ incident to three line segments. All the cut locus vertices are within Face 0. The cut locus has symmetry with respect to reflection about $x = 0$, a result of the path unfoldings that create the six copies of p having the same symmetry.

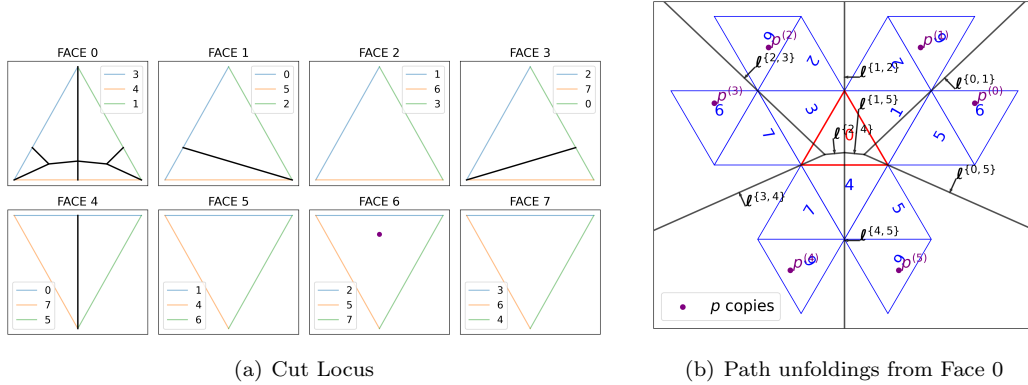


Figure 13: Cut locus of $p = (0, \frac{1}{2})$ chosen on a_1

Line c_1 : We find that the structure of the cut locus of $p \in c_1$ is isomorphic to Figure 14, with two vertices $x^{\{0,1,4,5\}}$ and $x^{\{1,2,3,4\}}$ both incident to four line segments. All the cut locus vertices are within Face 0. The

cut locus has symmetry with respect to reflection about the line $y = -\frac{x}{\sqrt{3}}$, a result of the path unfoldings that create the six copies of p having the same symmetry.

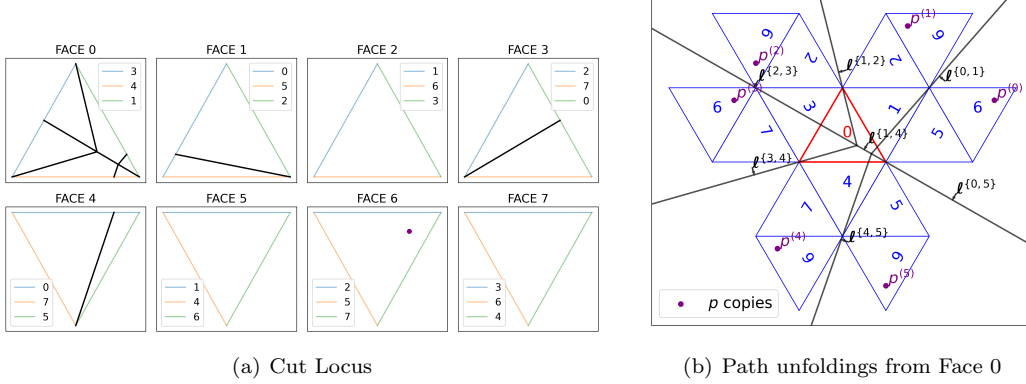


Figure 14: Cut locus of $p = (\frac{\sqrt{3}}{2}, \frac{1}{2})$ chosen on c_1

Region τ_1 : We find that the structure of the cut locus of $p \in \tau_1$ is isomorphic to Figure 15, with four vertices $x^{\{0,1,5\}}$, $x^{\{2,3,4\}}$, $x^{\{1,2,4\}}$, and $x^{\{1,4,5\}}$ incident to three line segments. All cut locus vertices are within Face 0.

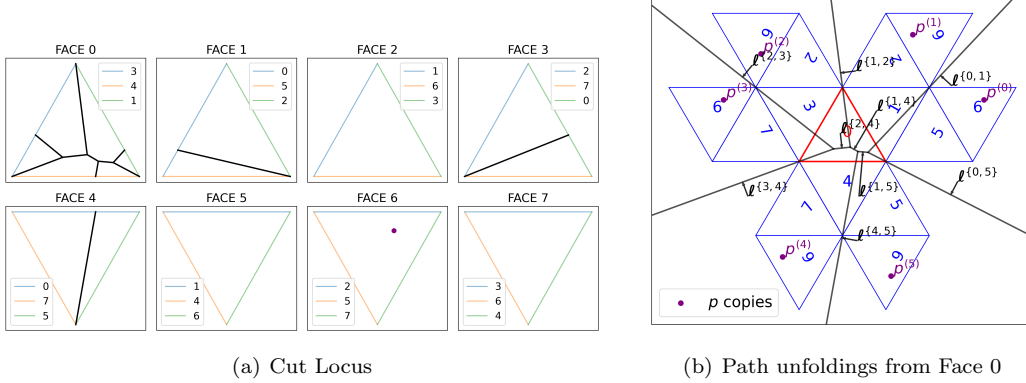


Figure 15: Cut locus of $p = (0.45, 0.5)$ chosen in τ_1

We will now consider the cut locus of points in v_1 and e_1 , which are not on the interior of Face 6. Since v_1 is a point, we will simply use Algorithm 2 to calculate it. For e_1 , we use the same method as before, though with a different set of path unfoldings.

Point v_1 : From Algorithm 2, we find that the cut locus of p chosen on vertex v_1 is isomorphic to Figure 16. When viewed on the octahedron, this structure is a star, with four lines joined at a point. The point is the antipode of p , and the four lines are the four octahedron edges incident to antipode.

Line e_1 : We claim that the cut locus of $p \in e_1$ is isomorphic to Figure 17. Due to the reflective symmetry

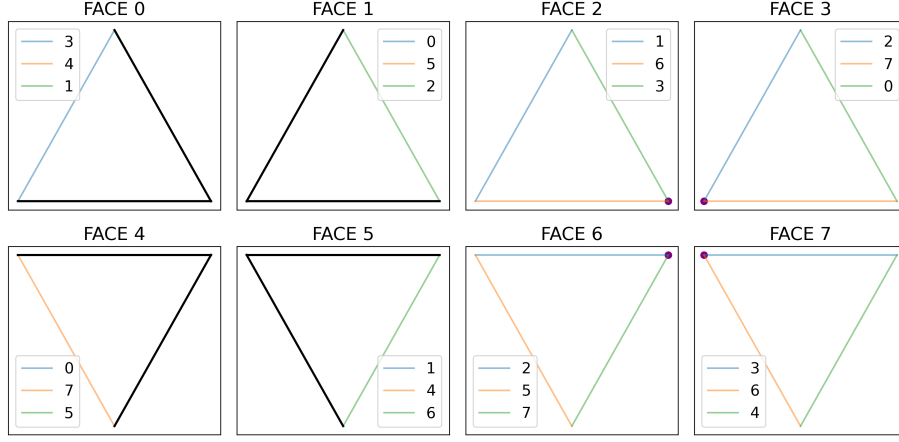


Figure 16: Cut locus of p chosen to be v_1

about the equator of an octahedron, we have that the cut loci on Faces 0, 1, 2, and 3 are symmetric to the cut loci on Faces 4, 5, 6, and 7 respectively. We find that the cut loci on Faces 1, 3, 5, and 7 arise from two copies of p . From the symmetry of the two copies, the cut locus is exactly one edge of each of these faces. We display the case for Face 1 in Figure 18(b).

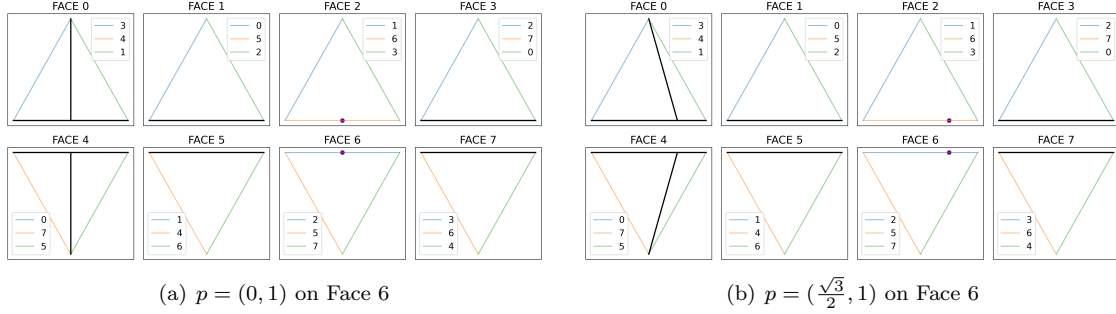


Figure 17: Cut locus of p chosen on e_1

For Face 0, we find that the cut locus arises from four copies of p , labeled⁴ $p^{(1)}, p^{(2)}, p^{(4)}, p^{(5)}$ as in Figure 18(a). The cut locus is isomorphic to a star, with four line segments joined at a vertex $x^{\{1,2,4,5\}}$, the antipode of p on the octahedron. Three of the line segments occur on Face 0, while $\ell^{\{4,5\}}$ only occurs on Face 4. The cut locus has symmetry with respect to reflection about the bottom edge of Face 0, a result of the path unfoldings that create the four copies of p having the same symmetry. Additionally, when p is at the midpoint of edge e_1 , the cut locus has symmetry with respect to reflection about $x = 0$.

⁴We choose this so that each point arises from the same path unfolding as the point with the same label earlier in this section.

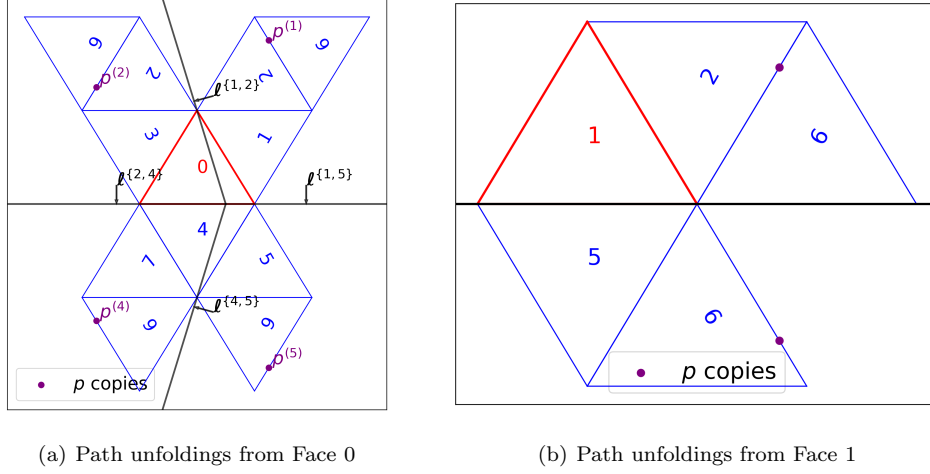


Figure 18: Copies of p and path unfoldings that contribute to the cut locus of $p = (\frac{\sqrt{3}}{2}, 1)$ chosen on e_1

3.3.2 Geodesic Complexity Lower Bound

We will use Theorem 2.1 to show that $GX(X) \geq 4$ (needing at least 5 sets in any geodesic motion planner). To do this, we will embed simplices into $X \times X$ using Lemma 2.1. Let $p^* = o_6$, and $q^* = (0, 0)$ on Face

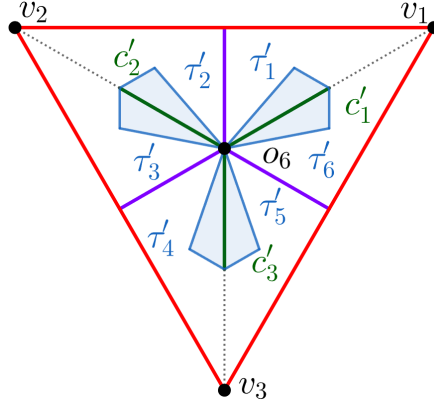


Figure 19: Constructed τ'_i and c'_i . Endpoints are the midpoint of each c_i and the centroid of each τ_i .

0, and let $\tau'_i \subseteq \overline{\tau_i}$ and $c'_i \subseteq \overline{c_i}$ be closed sets constructed as in Figure 19. Let $p_{c_1}, p_{c_2}, p_{c_3} : [0, 1] \hookrightarrow X$ be embeddings such that $\text{img}(p_{c_i}) = c'_i$ and $p_{c_i}(1) = o_6 = p^*$. Similarly, let $p_{\tau_1}, \dots, p_{\tau_6} : T_2 \hookrightarrow X$ be embeddings such that $\text{img}(p_{\tau_i}) = \tau'_i$ and $p_{\tau_i}(S_2) = c'_j$ for c'_j bordering τ'_i . (Recall from Definition 2.3 that T_2 is a triangle in $\mathbb{R}^2$ and $S_2 \subseteq T_2$ is one of its edges.)

Let $\mathcal{J} := \{J \subseteq \{0, \dots, 5\} : |J| = 3\}$, and let $\mathcal{I} := \{\{i, j\} \subseteq \{0, \dots, 5\} : j \equiv i + 1 \pmod{6}\}$ be collections of subsets of $\{0, \dots, 5\}$. We will define $q^{\tau_i, A, J} : T_2 \hookrightarrow X$ analogously to the construction in Section 3.2.2.

Explicitly, for $A \in \mathcal{I} \cup \mathcal{J}$, $J \in \mathcal{J}$, and $|A \cap J| = 2$, $q^{\tau_i, A, J}$ varies linearly between x^A and x^J via $q^{\tau_i, A, J}(\mathbf{t}) = (1 - \Sigma \mathbf{t}) \cdot x^A(p_{\tau_i}(\mathbf{t})) + (\Sigma \mathbf{t}) \cdot x^J(p_{\tau_i}(\mathbf{t}))$, where $\Sigma \mathbf{t}$ is the sum $\sum \mathbf{t}_i$. Since x^A and x^J are on line $\ell^{A \cap J}$, $q^{\tau_i, A, J}$ is also on $\ell^{A \cap J}$. Additionally, for $\mathbf{t} \in S_2$, we have $q^{\tau_i, A, J}(\mathbf{t}) = x^J(p_{\tau_i}(\mathbf{t}))$. We display $q^{\tau_1, \{0,5\}, \{0,1,5\}}$, $q^{\tau_1, \{1,2,4\}, \{1,4,5\}}$, and relevant x^A for a few selections of $p \in \tau'_1$ in Figure 20.

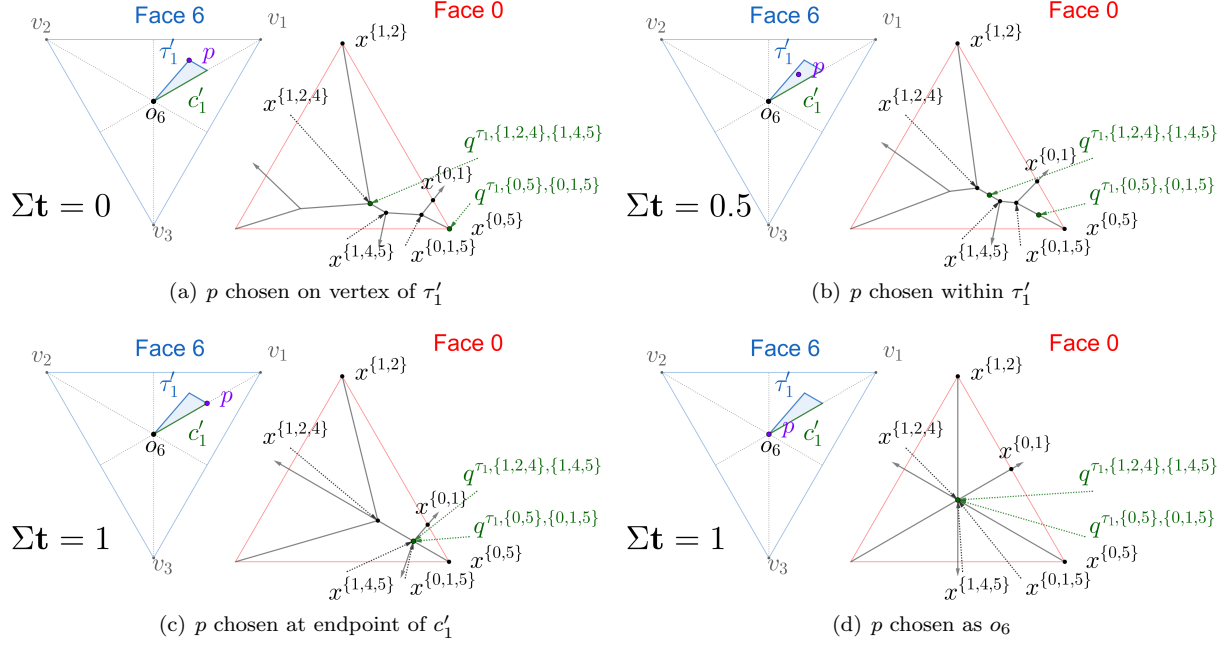


Figure 20: Defined points on Face 0 as p varies about τ'_1

We will now describe our $\mathcal{F}_i$ to use Theorem 2.1 on the octahedron:

- $\mathcal{F}_0 = \{\Delta_0 \mapsto (p^*, q^*)\}$.
- $\mathcal{F}_1$ will be defined by three lines. Consider p_{c_1} and the map $x^{\{0,1,4,5\}} \circ p_{c_1}$. By Lemma 2.1, this describes an embedding $f: \Delta_1 \hookrightarrow X \times X$ where $\pi_1(f(t, 1-t)) = p_{c_1}(t)$ and $\pi_2(f(t, 1-t)) = x^{\{0,1,4,5\}}(p_{c_1}(t))$. This map follows $(p, x^{\{0,1,4,5\}}(p))$ as p varies across c'_1 . We will construct symmetric embeddings using maps p_{c_2} and $x^{\{2,3,4,5\}} \circ p_{c_2}$, and maps p_{c_3} and $x^{\{0,1,2,3\}} \circ p_{c_3}$.
- $\mathcal{F}_2$ will be defined by twelve simplices. We will vary p along each of the τ'_i , and choose q to be one of the two vertices of the cut locus whose limit is the vertex chosen in $\mathcal{F}_1$ as p approaches c'_i . For the case of p_{τ_1} , these vertices will be $x^{\{0,1,5\}}$ and $x^{\{1,4,5\}}$. Explicitly, consider the maps p_{τ_1} and $x^{\{0,1,5\}} \circ p_{\tau_1}$. By Lemma 2.1, this describes an embedding $f: \Delta_2 \hookrightarrow X \times X$ where for $\mathbf{t} \in T_2$, $\pi_1(f(\mathbf{t}, 1 - \Sigma \mathbf{t})) = p_{\tau_1}(\mathbf{t})$ and $\pi_2(f(\mathbf{t}, 1 - \Sigma \mathbf{t})) = x^{\{0,1,5\}}(p_{\tau_1}(\mathbf{t}))$. This map follows $(p, x^{\{0,1,5\}}(p))$ as p varies around τ'_1 (see Figure 15). We will construct an embedding similarly from the maps p_{τ_1} and $x^{\{1,4,5\}} \circ p_{\tau_1}$. We construct the ten remaining embeddings symmetrically from the five other τ_i .
- $\mathcal{F}_3$ will be defined by 30 simplices, five for each τ_i , corresponding to the five cut locus lines incident to the cut locus vertices used in $\mathcal{F}_2$. Consider p_{τ_1} and the map $q^{\tau_1, \{0,1\}, \{0,1,5\}} \boxtimes (x^{\{0,1,5\}} \circ p_{\tau_1})$. Since when $\Sigma \mathbf{t} = 1$, we have $q^{\tau_1, \{0,1\}, \{0,1,5\}}(\mathbf{t}) = x^{\{0,1,5\}}(p_{\tau_1}(\mathbf{t}))$, and these points are otherwise distinct, we have by Lemma 2.1 that this defines an embedding $f: \Delta_3 \hookrightarrow X \times X$ whose image looks like (p, q) for $p \in \tau'_1$ and $q \in \ell^{\{0,1\}}(p)$. We will define two similar embeddings with p_{τ_1} using the maps

$q^{\tau_1, \{0,5\}, \{0,1,5\}} \boxtimes (x^{\{0,1,5\}} \circ p_{\tau_1})$ and $q^{\tau_1, \{4,5\}, \{1,4,5\}} \boxtimes (x^{\{1,4,5\}} \circ p_{\tau_1})$.

For another embedding, consider p_{τ_1} and the map $(x^{\{0,1,5\}} \circ p_{\tau_1}) \boxtimes (x^{\{1,4,5\}} \circ p_{\tau_1})$. When $\Sigma \mathbf{t} = 1$, we have $x^{\{0,1,5\}}(p_{\tau_1}(\mathbf{t})) = x^{\{1,4,5\}}(p_{\tau_1}(\mathbf{t}))$ since $p_{\tau_1}(\mathbf{t}) \in \bar{c}_1$. These points are otherwise distinct, so this defines an embedding $f: \Delta_3 \hookrightarrow X \times X$ whose image looks like (p, q) for $p \in \tau_1'$ and $q \in \ell^{\{1,5\}}(p)$.

Finally, consider p_{τ_1} and the map $(x^{\{1,4,5\}} \circ p_{\tau_1}) \boxtimes q^{\tau_1, \{1,2,4\}, \{1,4,5\}}$. Since when $\Sigma \mathbf{t} = 1$, we have $x^{\{1,4,5\}}(p_{\tau_1}(\mathbf{t})) = q^{\tau_1, \{1,2,4\}, \{1,4,5\}}(\mathbf{t})$, and these points are otherwise distinct, this defines an embedding $f: \Delta_3 \hookrightarrow X \times X$ whose image looks like (p, q) for $p \in \tau_1'$ and $q \in \ell^{\{1,4\}}(p)$.

This describes the five embeddings corresponding to τ_1 . We construct five symmetric embeddings for each of the five remaining τ_i .

- $\mathcal{F}_4$ will be defined by 36 simplices, six for each τ_i , corresponding to the three cut locus Voronoi cells incident to each cut locus vertex chosen in $\mathcal{F}_2$. We will inspect τ_1 (with cut locus structure as in Figure 15) and the vertex $x^{\{0,1,5\}}$. Consider p_{τ_1} and the map $(x^{\{0,1,5\}} \circ p_{\tau_1}) \boxtimes q^{\tau_1, \{0,1\}, \{0,1,5\}} \boxtimes (x^{\{1,4,5\}} \circ p_{\tau_1})$. When $\Sigma \mathbf{t} = 1$, we have $x^{\{0,1,5\}}(p_{\tau_1}(\mathbf{t})) = x^{\{1,4,5\}}(p_{\tau_1}(\mathbf{t})) = q^{\tau_1, \{0,1\}, \{0,1,5\}}(\mathbf{t})$, and these points are otherwise not collinear. Thus, by Lemma 2.1, this defines an embedding $f: \Delta_4 \hookrightarrow X \times X$ where for $\mathbf{d} \in \Delta_4$, $f(\mathbf{d}) = (p, q)$ for $p \in \tau_1'$ and q in the convex hull of $x^{\{0,1,5\}}(p)$, $x^{\{1,4,5\}}(p)$, and a point on $\ell^{\{0,1\}}$. We will create two similar embeddings by considering the maps $(x^{\{0,1,5\}} \circ p_{\tau_1}) \boxtimes (x^{\{1,4,5\}} \circ p_{\tau_1}) \boxtimes q^{\tau_1, \{0,5\}, \{0,1,5\}}$ and $(x^{\{0,1,5\}} \circ p_{\tau_1}) \boxtimes q^{\tau_1, \{0,5\}, \{0,1,5\}} \boxtimes q^{\tau_1, \{0,1\}, \{0,1,5\}}$.

We will now inspect the other vertex, $x^{\{1,4,5\}}$. We will create three similar embeddings by considering the map $(x^{\{1,4,5\}} \circ p_{\tau_1}) \boxtimes (x^{\{0,1,5\}} \circ p_{\tau_1}) \boxtimes q^{\tau_1, \{1,2,4\}, \{1,4,5\}}$, the map $(x^{\{1,4,5\}} \circ p_{\tau_1}) \boxtimes q^{\tau_1, \{1,2,4\}, \{1,4,5\}} \boxtimes q^{\tau_1, \{4,5\}, \{1,4,5\}}$, and the map $(x^{\{1,4,5\}} \circ p_{\tau_1}) \boxtimes q^{\tau_1, \{4,5\}, \{1,4,5\}} \boxtimes (x^{\{0,1,5\}} \circ p_{\tau_1})$.

This describes the six embeddings corresponding to τ_1 . We construct six symmetric embeddings for each of the five remaining τ_i .

We visualize a few examples of sets in $\mathcal{F}_2$, $\mathcal{F}_3$, and $\mathcal{F}_4$ in Figure 21.

We will verify the properties of Theorem 2.1 for a few embeddings and geodesic choices. We assert that similar arguments extend to all embeddings by symmetry in their construction. First, each $\mathcal{F}_i$ contains the embeddings of $\Delta_i \hookrightarrow X \times X$ by construction, so property (a) always holds.

- Consider $F_0 \in \mathcal{F}_0$. Property (b) is trivial since $\text{img}(F_0) = \{(p^*, q^*)\}$ is a single point. We will consider a GMPR Γ_0 on $\text{img}(F_0)$. We have six options, determined by which copy of p^* ($p^{(0)}, \dots, p^{(5)}$) we connect q^* to. Assume our choice connects q^* to $p^{(3)}$. Then our choice for $F_1 \in \mathcal{F}_1$ will be the embedding constructed from maps p_{c_1} and $x^{\{0,1,4,5\}} \circ p_{c_1}$. (i) is trivially satisfied. (ii) is satisfied since for any element $(p, q) \in \text{relint}^*(F_1)$, q is a point on the cut locus of p with geodesics only to $p^{(0)}$, $p^{(1)}$, $p^{(4)}$, and $p^{(5)}$. (iii) is satisfied since there are four GMPRs on $\text{img}(F_1)$, each distinguished by which of $p^{(0)}$, $p^{(1)}$, $p^{(4)}$, or $p^{(5)}$ they choose to connect to. Thus, property (c) is satisfied.
- Consider $F_1 \in \mathcal{F}_1$. Property (b) is satisfied since any geodesic for $(p, q) \in \text{relint}^*(F_1)$, defined by connecting q to one of $p^{(0)}$, $p^{(1)}$, $p^{(4)}$, or $p^{(5)}$, extends to a GMPR distinguished by the same choice. We will consider one of the four GMPRs Γ_1 . We will choose Γ_1 to connect to $p^{(4)}$. Then our choice for $F_2 \in \mathcal{F}_2$ will be the embedding constructed from maps p_{τ_1} and $x^{\{0,1,5\}} \circ p_{\tau_1}$. This embedding follows $x^{\{0,1,5\}}(p)$ as p varies across τ_1' . (i) is satisfied by construction. (ii) is satisfied since for any element $(p, q) \in \text{relint}^*(F_2)$, q is a point on the cut locus of p with geodesics only to $p^{(0)}$, $p^{(1)}$, and $p^{(5)}$. (iii) is satisfied since there are three GMPRs on $\text{img}(F_2)$, each distinguished by which of $p^{(0)}$, $p^{(1)}$, or $p^{(5)}$ they choose to connect to. Thus, property (c) is satisfied.
- Consider $F_2 \in \mathcal{F}_2$. Property (b) is satisfied since any geodesic for $(p, q) \in \text{relint}^*(F_2)$, defined by connecting q to one of $p^{(0)}$, $p^{(1)}$, or $p^{(5)}$, extends to a GMPR distinguished by the same choice.

We will consider one of the three GMPRs Γ_2 . We will choose Γ_2 to connect to $p^{(5)}$.

Then our choice for $F_3 \in \mathcal{F}_3$ will be the embedding constructed from maps p_{τ_1} and $q^{\tau_1, \{0,1\}, \{0,1,5\}} \boxtimes (x^{\{0,1,5\}} \circ p_{\tau_1})$. This embedding follows a segment of $\ell^{\{0,1\}}(p)$ for each choice of p .

(i) is satisfied by construction. (ii) is satisfied since for any element $(p, q) \in \text{relint}^*(F_3)$, q is a point on the cut locus of p with geodesics only to $p^{(0)}$ and $p^{(1)}$. (iii) is satisfied since there are two GMPRs on $\text{img}(F_2)$, each distinguished by which of $p^{(0)}$ or $p^{(1)}$ they choose to connect to. Thus, property (c) is satisfied.

- Consider $F_3 \in \mathcal{F}_3$. Property (b) is satisfied since any geodesic for $(p, q) \in \text{relint}^*(F_3)$, defined by connecting q to one of $p^{(0)}$ or $p^{(1)}$, extends to a GMPR distinguished by the same choice.

We will consider one of the two GMPRs Γ_3 . We will choose Γ_3 to connect to $p^{(0)}$.

Then our choice for $F_4 \in \mathcal{F}_4$ will be the embedding constructed from maps p_{τ_1} and $(x^{\{0,1,5\}} \circ p_{\tau_1}) \boxtimes q^{\tau_1, \{0,1\}, \{0,1,5\}} \boxtimes (x^{\{1,4,5\}} \circ p_{\tau_1})$. This embedding follows a region bounded partially by $\ell^{\{0,1\}}(p)$ and $\ell^{\{1,5\}}(p)$ for each choice of p .

(i) is satisfied by construction. (ii) is satisfied since for any element $(p, q) \in \text{relint}^*(F_4)$, q is a point on the cut locus of p with a geodesic only to $p^{(1)}$. (iii) is satisfied since there is only one GMPR on $\text{img}(F_4)$, as the only choice that extends to the interior is to go to $p^{(1)}$. Thus, property (c) is satisfied.

- Consider $F_4 \in \mathcal{F}_4$. Property (b) is satisfied since any geodesic for $(p, q) \in \text{relint}^*(F_4)$ is defined by connecting q to $p^{(1)}$, and extends to the only GMPR over $\text{img}(F_4)$.

We do not need to check (c) as 4 is our maximal dimension.

Then by Theorem 2.1, $GC(X) \geq 4$ (needing at least 5 sets in any geodesic motion planner).

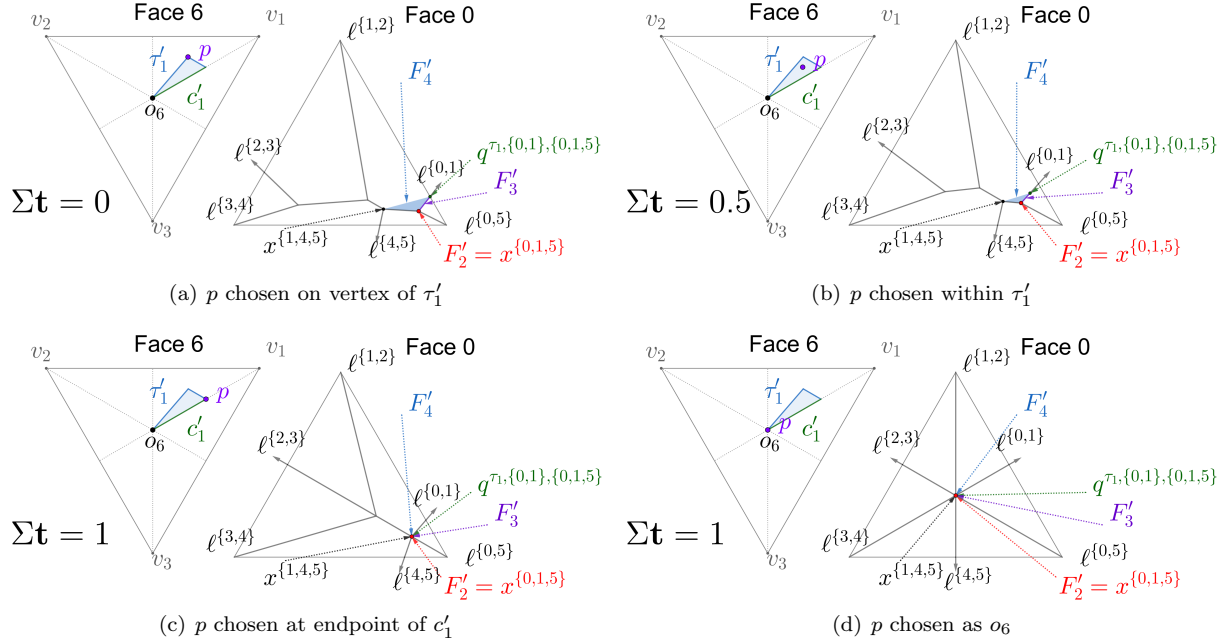


Figure 21: Sets F'_i as p varies about τ'_1

We visualize this similar to the method in Figure 10, with the addition of one dimension. For $i > 1$,

each F_i is a map $\Delta_i \hookrightarrow X \times X$, and is generated using Lemma 2.1 with $k = 2$. Recall that in this case, T_2 is a triangle in $\mathbb{R}^2$, and for $\mathbf{t} \in T_2$, $D_{\mathbf{t},i-1} = \{\mathbf{x} \in \Delta_i : (\mathbf{x}_1, \mathbf{x}_2) = \mathbf{t}\}$. From construction, we have that the projection onto the first factor $\pi_1(F_i(D_{\mathbf{t},i-1}))$ is a single point, and the image on the second factor $\pi_2(F_i(D_{\mathbf{t},i-1}))$ is the convex hull of $i - 1$ points. Let this hull be F'_i . In Figure 21, we plot the F'_i corresponding to the F_i we inspected for a few values of $\mathbf{t}$. We also plot the first factor of each F_i , which is always a point $p \in \tau'_1$.

3.3.3 Explicit Geodesic Motion Planner

We will construct an explicit geodesic motion planner on five sets to show that $GC(X) \leq 4$. We take inspiration from the work of Davis on the cube [6] and construct the sets mostly based on **multiplicity**, the number of distinct geodesics between a pair of points. From the possible cut loci in Section 3.3.1, the possible multiplicities are 1, 2, 3, 4, and 6, since no pair of points (p, q) has q in exactly five cells of the cut locus of p .⁵

We extend the enumeration of the regions in Figure 11 to the full octahedron. In total, the octahedron is partitioned into eight points o_i , six points v_i , twelve edges e_i , twenty-four line segments a_i , twenty-four line segments c_i , and forty-eight cells τ_i . We partition $E_1 \sqcup \dots \sqcup E_5 = X \times X$ in the following way:

- Let E_5 be the set of all pairs in $X \times X$ such that each point is either a vertex, a midpoint of an edge, or the center of a face. Since there are six vertices, twelve edges, and eight faces in the octahedron, E_5 is a collection of 26^2 points in $X \times X$. All pairs of points with multiplicity 6 are in E_5 .
- Let M_1 be the set of all pairs in $X \times X$ with multiplicity 1, and let $E_1 := M_1 \setminus E_5$.
- Let M_4 be the set of all pairs in $X \times X$ with multiplicity 4, and let $E_4 := M_4 \setminus E_5$.
- Let M_3 be the set of all pairs in $X \times X$ with multiplicity 3.

We additionally consider the following elements with multiplicity 2:

For each a_i , the cut locus of $p \in a_i$ has exactly two vertices of degree three. For each vertex, two of the three incident lines have an endpoint at octahedron vertices. The specified vertices and lines are limits of vertices and lines in the cut locus of p chosen in the neighboring τ_j or τ_k . We will include (p, q) for $p \in a_i \cup \tau_j \cup \tau_k$ and q on one of these four cut locus lines.

In the example of a_1 and τ_1 in Figures 13 and 15, we consider lines $\ell^{\{0,1\}}$, $\ell^{\{0,5\}}$, $\ell^{\{2,3\}}$, and $\ell^{\{3,4\}}$ on Face 0, and their extensions on Faces 1 and 3. We visualize this in Figure 22.

Formally, we construct set S : For each a_i with neighboring τ_j and τ_k , we include all (p, q) where $p \in a_i \cup \tau_j \cup \tau_k$ and q is on one of the four specified lines of the cut locus.

We will set $E_3 := (M_3 \cup S) \setminus E_5$.

- Let M_2 be the set of all pairs in $X \times X$ with multiplicity 2. We will set $E_2 := M_2 \setminus (E_5 \cup E_3)$.

We illustrate the points of multiplicity 2 in set E_3 . For $p \in X$, we define $E'_i := \{q \in X : (p, q) \in E_i\}$. In Figure 22, we depict the sets E'_2 , E'_3 , and E'_4 on Face 0 for p chosen in τ_1 and a_1 . In particular, points on cut locus lines $\ell^{\{0,1\}}$, $\ell^{\{0,5\}}$, $\ell^{\{2,3\}}$, and $\ell^{\{3,4\}}$ have multiplicity 2 with p , but have been included in set E'_3 .

Lemma 3.1. *For X compact Hausdorff and $A, B \subseteq X$, let $f: A \rightarrow X$ and $g: \overline{B} \rightarrow X$ be maps such that for all $x \in A \cap \overline{B}$, $f(x) \neq g(x)$. Then an open set separates $S_A := \{(a, f(a)) : a \in A\}$ from $S_B := \{(b, g(b)) : b \in \overline{B}\}$.*

⁵The multiplicity of (p, q) on a convex polyhedron is the number of Voronoi cells from Algorithm 2 that q is in. By Lemma 2.7, we have a geodesic by connecting the image of q to the copy of p associated with the cell, and Lemma 2.4 indicates this accounts for all geodesics.

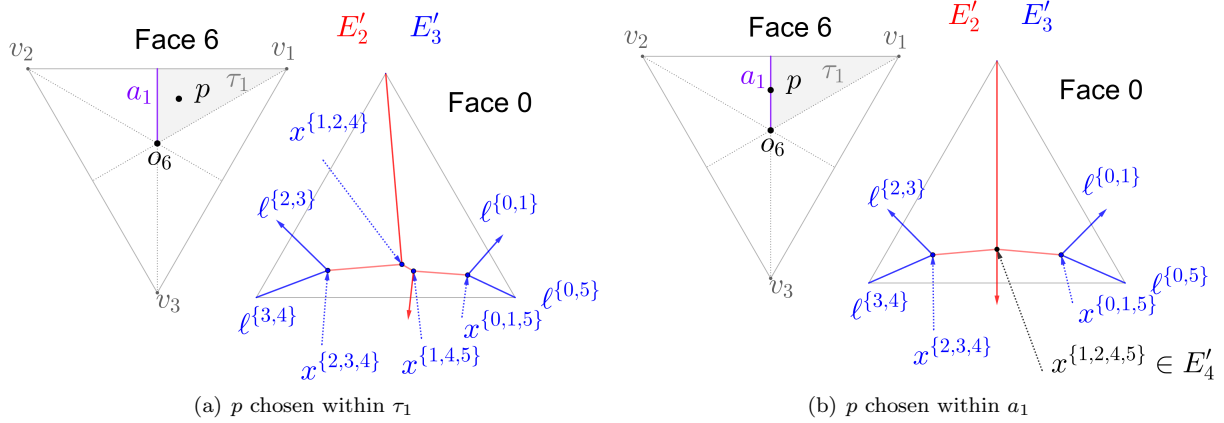


Figure 22: Sets E'_2 , E'_3 , and E'_4 for $p \in \tau_1 \cup a_1$

Since X is compact, $\overline{B}$ is compact, and since X is Hausdorff, $X \times X$ is Hausdorff. The map $b \mapsto (b, g(b))$ is continuous from continuity of g . Since S_B is the image of $\overline{B}$ under this map, it is compact and thus closed. From our hypotheses, S_A is disjoint from S_B . Then $(S_B)^C$ is open and separates S_A from S_B . ■

We will now construct a GMPR $\sigma_i: E_i \rightarrow GX$ for each $1 \leq i \leq 5$. To construct some of these GMPRs, we use Lemma 3.1 to split E_i into a **clopen partition**.⁶

Sets E_1 and E_5 : There is only one choice for $\sigma_1: E_1 \rightarrow GX$, as there is a unique geodesic between each $(p, q) \in E_1$. Continuity follows from Lemma 3.12 of Chapter 1 in [2]. For $\sigma_5: E_5 \rightarrow GX$, we arbitrarily select a geodesic for each element of E_5 . This is continuous since E_5 is discrete.

Set E_4 : The points that have multiplicity 4 are (p, q) such that p is on some a_i , c_i , v_i , or e_i . Given p , the cut locus is isomorphic to those in Figures 13, 14, 16, or 17 respectively, and q must be a cut locus vertex with degree four. For $p \in c_i$, there are two choices for q , as both cut locus vertices have degree four. For p chosen on some e_i , v_i , or a_i , there is only one choice for q . We may ignore (p, q) with multiplicity 4 where p is some v_i or on the midpoint of some e_i , as in this case $(p, q) \in E_5$. Thus, we enumerate line segments e'_j , where removing the midpoint from each e_i creates two e'_j . Then for $(p, q) \in E_4$, p is on some a_i , c_i , or e'_j , and these line segments are pairwise separated from each other.

Clopen Partition: To create a GMPR, we partition E_4 into the following types of sets:

- Let L be either some a_i or some e'_i , and define $f: \overline{L} \rightarrow X$ in the following way: For any point $p \in L$ there is a unique vertex of the cut locus with degree four. Let $f(p)$ be this point, and note that $f(p)$ varies continuously for $p \in L$. An endpoint of L is either on some edge e_i , some v_i or some o_i . As p approaches an endpoint, the limit $f(p)$ is well defined. In all cases, the limit will be the (only) vertex of the cut locus of p . We will define $f(p)$ as this limit. We construct a set $\{(p, f(p)) : p \in L\}$. We additionally note that f is well defined and continuous on $\overline{L}$. There are forty-eight sets of this type, arising from the twenty-four choices of a_i and twelve choices of e_j (each edge e_j creates two e'_i).
- Let L be some c_i , and define $f: \overline{L} \rightarrow X$ in the following way: For $p \in L$, there are two vertices of

⁶Where each set is clopen - both closed and open.

the cut locus with degree four, and both vary continuously with p . Pick one and let this be $f(p)$ for $p \in L$. An endpoint of L is either some v_i or some o_i . As p approaches an endpoint, the limit $f(p)$ is well defined as the (only) vertex of the cut locus of p . We will define $f(p)$ as this limit for p an endpoint of L .

We construct sets $\{(p, f(p)) : p \in L\}$. We additionally note that f is well defined and continuous on $\bar{L}$. There are forty-eight sets of this type, as there are two choices of f for each of the twenty-four choices of c_i .

Each pair of sets satisfies the hypotheses for Lemma 3.1. For most sets this is trivial, as for L_1, L_2 each some a_i, c_i, v_i , or e'_j , if $L_1 \neq L_2$ then $L_1 \cap \bar{L}_2 = \emptyset$. The only case we must consider is if $L_1 = L_2$, which can only yield two different sets for $L_1 = c_i$. This case satisfies the hypotheses since $c_i \cap \bar{c}_i = c_i$ and for any $p \in c_i$, the two vertices of the cut locus of p are distinct. Thus, this collection of sets is a clopen partition of E_4 . To define a GMPR on all of E_4 , it is enough to define a GMPR on each of these ninety-six sets.

Geodesic Motion Planning Rule: Consider any of these sets S . We claim that there are exactly four GMPRs on S . For $(p, q) \in S$, q is adjacent to exactly four cells in the cut locus of p by construction. These cells vary continuously wrt. p , as they arise from a Voronoi diagram generated by copies of p . The four GMPRs are defined by which copy of p we choose to create a path to.

Consider the case of a_1 , displayed in Figure 13. In this case, q is the point equidistant from copies $p^{(1)}$, $p^{(2)}$, $p^{(4)}$, and $p^{(5)}$. We will define a GMPR by choosing one of these copies of p arbitrarily, and letting $\gamma: [0, 1] \rightarrow \mathbb{R}^2$ be the path from this copy to the image of q . With f the path unfolding of the chosen copy of p , $f^{-1} \circ \gamma$ defines a path from p to q in the octahedron.⁷ Continuity of this path follows from continuity of the locations of the copies of p . For each of the sets composing E_4 , we arbitrarily choose one of the four GMPRs. Our GMPR σ_4 is the piecewise function with our ninety-six chosen GMPRs as components.

Set E_3 : The points that have multiplicity 3 are (p, q) such that p is either on some τ_i or some a_i . The cut locus of p will be isomorphic to those in Figures 15 and 13 respectively. If p is on some τ_i , q must be one of the four vertices of the resulting cut locus. If p is on some a_i , q must be one of the two vertices of the resulting cut locus with degree three. We additionally must consider the points of multiplicity 2 that we added to E_3 . These points are (p, q) such that p is on some connected $a_i \cup \tau_j \cup \tau_k$, and q is on one of the four specified cut locus lines.

Clopen Partition: We partition E_3 into the following types of sets:

- Consider some a_i and its adjacent sets τ_j, τ_k , and define $A := a_i \cup \tau_j \cup \tau_k$. For $p \in A$, there are two vertices of the cut locus that always have degree three. These vertices vary continuously with p , as they always arise from the same copies of p throughout A . Two of the cut locus lines connected to each vertex are also included in set E_3 . In the case of a_1 and τ_1 (displayed in Figure 22), we inspect vertex $x^{\{0,1,5\}}$ with lines $\ell^{\{0,1\}}$ and $\ell^{\{0,5\}}$, and vertex $x^{\{2,3,4\}}$ with lines $\ell^{\{2,3\}}$ and $\ell^{\{3,4\}}$.

We pick one of these two and construct a set S of all (p, q) with $p \in A$ and q either the vertex itself, or on one of the two specified cut locus lines connected to the vertex. There are forty-eight of these sets, as we construct two for each of the twenty-four a_i .

- Consider some τ_i , and note that for p chosen on τ_i , there are a total of four vertices in the cut locus of p . We have already included two of these vertices in sets of the first type, so we will consider the other two. In the case of Figure 22(a), we consider the vertices $x^{\{1,2,4\}}$ and $x^{\{1,4,5\}}$.

As before, we will choose one vertex and define map $f: \bar{\tau}_i \rightarrow X$. Within τ_i , f will be the specified

⁷While f itself does not necessarily have an inverse, we may decompose γ so that each segment is only on one face of the path unfolding. Since f restricted to each face is invertible, we may use it to pull back each segment of γ into a path on X .

vertex. As p approaches the boundary of τ_i , each vertex has a well defined limit. As p approaches some c_i , $f(p)$ approaches a vertex of the resulting cut locus. In this case, which vertex $f(p)$ approaches depends on our initial choice of cut locus vertex to follow. In all other cases, as p approaches the boundary, $f(p)$ approaches the unique vertex of the cut locus with maximal degree. We construct set $\{(p, f(p)) : p \in \tau_i\}$ and note that f is well defined and continuous on $\bar{\tau}_i$. There are ninety-six of these sets, as we construct two for each of the forty-eight τ_i .

We will show that these sets form a clopen partition of E_3 . Let S_1 and S_2 be constructed sets. In most cases, S_1 and S_2 are easily separated on their first factors, so we will only consider when this is not true. We will also use the observation that in a metric space, if S_1 is not separable from S_2 by an open set, then $S_1 \cap \overline{S_2}$ is nonempty, and there is a sequence $(p_n, q_n)_n \subseteq S_2$ such that $\lim_{n \rightarrow \infty} (p_n, q_n)$ exists and is in S_1 .

- Assume S_1 and S_2 are both sets of the second type. In this case, they satisfy the hypotheses for Lemma 3.1, and thus are separable by an open set.
- Assume S_1 and S_2 are both sets of the first type. We may assume they arise from the same a_i and adjacent τ_j, τ_k .

Assume for the sake of contradiction that S_1 is not separable from S_2 . Then find sequence $(p_n, q_n)_n \subseteq S_2$ whose limit is some $(p, q) \in S_1$. Then p and all p_n must be in $a_i \cup \tau_j \cup \tau_k$. However, the cut locus of a point p' chosen in this region arises from a Voronoi diagram of points that vary continuously with p' . In particular, since each q_n is on particular lines of the cut locus of p_n , q must be on the closure of those same cut locus lines with respect to p . However, the cut locus lines that q_n can be on must be different from the cut locus lines that q can be on, from construction of S_1 and S_2 . The closures of these lines never intersect, which gives us the contradiction that $\lim_{n \rightarrow \infty} (p_n, q_n) \neq (p, q)$. In an example using τ_1 and a_1 , assume S_1 is created with vertex $x^{\{2,3,4\}}$ and lines $\ell^{\{2,3\}}$ and $\ell^{\{3,4\}}$, and S_2 is created with vertex $x^{\{0,1,5\}}$ and lines $\ell^{\{0,1\}}$ and $\ell^{\{0,5\}}$ (see Figure 22). We observe from our cut locus calculations that for no cut locus of a point $p' \in a_1 \cup \tau_1 \cup \tau_2$ does the set $\overline{\ell^{\{0,1\}}(p') \cup x^{\{0,1,5\}}(p') \cup \ell^{\{0,5\}}(p')}$ intersect $\ell^{\{2,3\}}(p') \cup x^{\{2,3,4\}}(p') \cup \ell^{\{3,4\}}(p')$.

- Assume S_1 is of the first type and S_2 is of the second type. We may assume S_1 arises from a_i and adjacent τ_j, τ_k , and S_2 arises from τ_j .

First, take $(p_n, q_n)_n \subseteq S_1$ such that the limit $(p, q) \in S_2$. This implies $p \in \tau_j$. Similar to the case where both sets were of the first type, we reach a contradiction since the closure of the cut locus lines that each q_n may be on do not intersect the cut locus vertices that q may be on. In an example using τ_1 and a_1 , for no cut locus of a point $p' \in \tau_1$ does the set $\overline{\ell^{\{0,1\}}(p') \cup x^{\{0,1,5\}}(p') \cup \ell^{\{0,5\}}(p')}$ intersect $x^{\{1,4,5\}}(p')$ or $x^{\{1,2,4\}}(p')$.

Now take $(p_n, q_n)_n \subseteq S_2$ such that the limit $(p, q) \in S_1$. From construction, $p_n \in \tau_j$ and $p \in a_i \cup \tau_j \cup \tau_k$. Since p must be the limit of the p_n , $p \in \bar{\tau}_j \cap (a_i \cup \tau_j \cup \tau_k)$, which implies either $p \in \tau_j$ or $p \in a_i$. By construction, q_n is a particular vertex of the cut locus of p_n . Since this vertex varies continuously for points on $\tau_j \cup a_i$, q must be the limit of this vertex as p_n approaches p . This results in a similar contradiction, as these vertices do not lie on any of the cut locus lines that q must be on due to $(p, q) \in S_1$. In an example using τ_1 and a_1 , for no cut locus of a point $p' \in \tau_1 \cup a_1$ do vertices $x^{\{1,4,5\}}(p')$, $x^{\{1,2,4\}}(p')$, or $x^{\{1,2,4,5\}}(p')$ intersect $\ell^{\{0,1\}}(p') \cup x^{\{0,1,5\}}(p') \cup \ell^{\{0,5\}}(p')$.

Thus, these 144 sets form a clopen partition of E_3 , and to define a GMPR over E_3 , it is enough to define a GMPR for each set.

Geodesic Motion Planning Rule: Each set of the second type contains (p, q) , where q is adjacent to three Voronoi cells of the cut locus of p . Similar to the case for E_4 , the set has three possible GMPRs, each corresponding to a choice of cell. We choose one arbitrarily for each of these 96 sets. Each set S of the

first type contains (p, q) for $p \in a_i \cup \tau_j \cup \tau_k$ for neighboring a_i, τ_j, τ_k , and q chosen as one of the resulting cut locus vertices, or on specified cut locus lines connected to the vertex. From construction, there is a single Voronoi cell of the cut locus of p that every choice of q is adjacent to. We will choose our GMPR to always approach the copy of p that creates this Voronoi cell. Our overall GMPR σ_3 is the piecewise function with the GMPR for each set as components. The continuity of each constituent GMPR is evident from the same argument as σ_4 , where we first construct a GMPR within the appropriate path unfolding on $\mathbb{R}^2$, then use f^{-1} to pull it into the octahedron.

Set E_2 : The points that have multiplicity 2 are a choice of $p \in X$ and q on a line of the cut locus of p .

Partition: We will partition E_2 into sets of two types.

- Consider some c_i , and its adjacent sets τ_j, τ_k , and define $A := c_i \cup \tau_j \cup \tau_k$. For $p \in c_i$, there is one cut locus line that does not touch any edge of the octahedron (for example, $\ell^{\{1,4\}}$ in Figure 14). When $p \in \tau_i$ or $p \in \tau_j$, there is one cut locus line that approaches our specified cut locus line as p approaches c_i . This line is always the middle of three cut locus lines that do not touch any edge of the octahedron, and in the example of Figure 15, we would consider $\ell^{\{1,4\}}$. We construct a set S of all (p, q) with $p \in A$ and q on the relative interior of this cut locus line. There are twenty-four of these sets, corresponding to one for each c_i .
- The remaining set is all other points in E_2 .

Geodesic Motion Planning Rule: Every point in E_2 is (p, q) with q on the relative interior of a line of the cut locus of p . Thus, to describe a GMPR σ_2 , it is enough to describe which side of the line σ_2 approaches q from. This is equivalent to constructing σ_2 to approach one of the two copies of p that generate the cut locus line. For each set of the first type, we arbitrarily choose which of the two copies of p the GMPR approaches from. For the set S of the second type, we will use the fact that the octahedron is orientable. This allows us to have a consistent notion of ‘clockwise’ for each point on the octahedron, which we use to choose a side of each line.

- For p chosen on some e_i, c_i , or o_i (e.g. Figure 23(a, b, c)), all lines of the cut locus that we must account for have a unique endpoint at a vertex of the octahedron. With respect to this endpoint, we have a notion of the clockwise side of each line. We construct σ_2 to approach each point q on a cut locus line from the clockwise side.
- For p chosen on some v_i (e.g. Figure 23(d)), the lines of the resulting cut locus have both of their endpoints at vertices of the octahedron. We can distinguish one vertex as the antipode of p , and we will use the vertex that is *not* the antipode to determine the clockwise side of each line. We construct σ_2 to approach each point on the cut locus line from the clockwise side.
- For p in some $A := a_i \cup \tau_j \cup \tau_k$ (with τ_i and τ_k adjacent to a_i , and edge e_ℓ being the edge adjacent to A), a few cut locus lines have an endpoint on vertices of the octahedron. We construct σ_2 to approach each point on these lines from the clockwise side (with respect to the octahedron vertex endpoint). For Figures 13 and 15, these lines are $\ell^{\{1,2\}}$ and $\ell^{\{4,5\}}$. There remain two lines not considered in other cases (one line is considered in the sets of the first kind, and four lines are considered in set E_3). As p approaches the (unique) adjacent edge e_ℓ , the limit of each of these lines is an octahedron edge. As before, we have a notion of the clockwise side of each cut locus line for $p \in e_\ell$, which allows us to define a clockwise side for the lines that approach these edges for $p \in A$. We construct σ_2 to approach each point on these cut locus lines from the clockwise side. In an example with τ_1, a_1 , and e_1 , lines $\ell^{\{1,5\}}$ and $\ell^{\{2,4\}}$ in Figures 13 and 15 approach the lines of the same name in Figure 18(a). We display the chosen directions for a_1 and τ_1 in Figure 23(e, f)

Continuity: We will show the continuity of σ_2 . Since we construct σ_2 similar to the GMPR of E_2 in [6], we

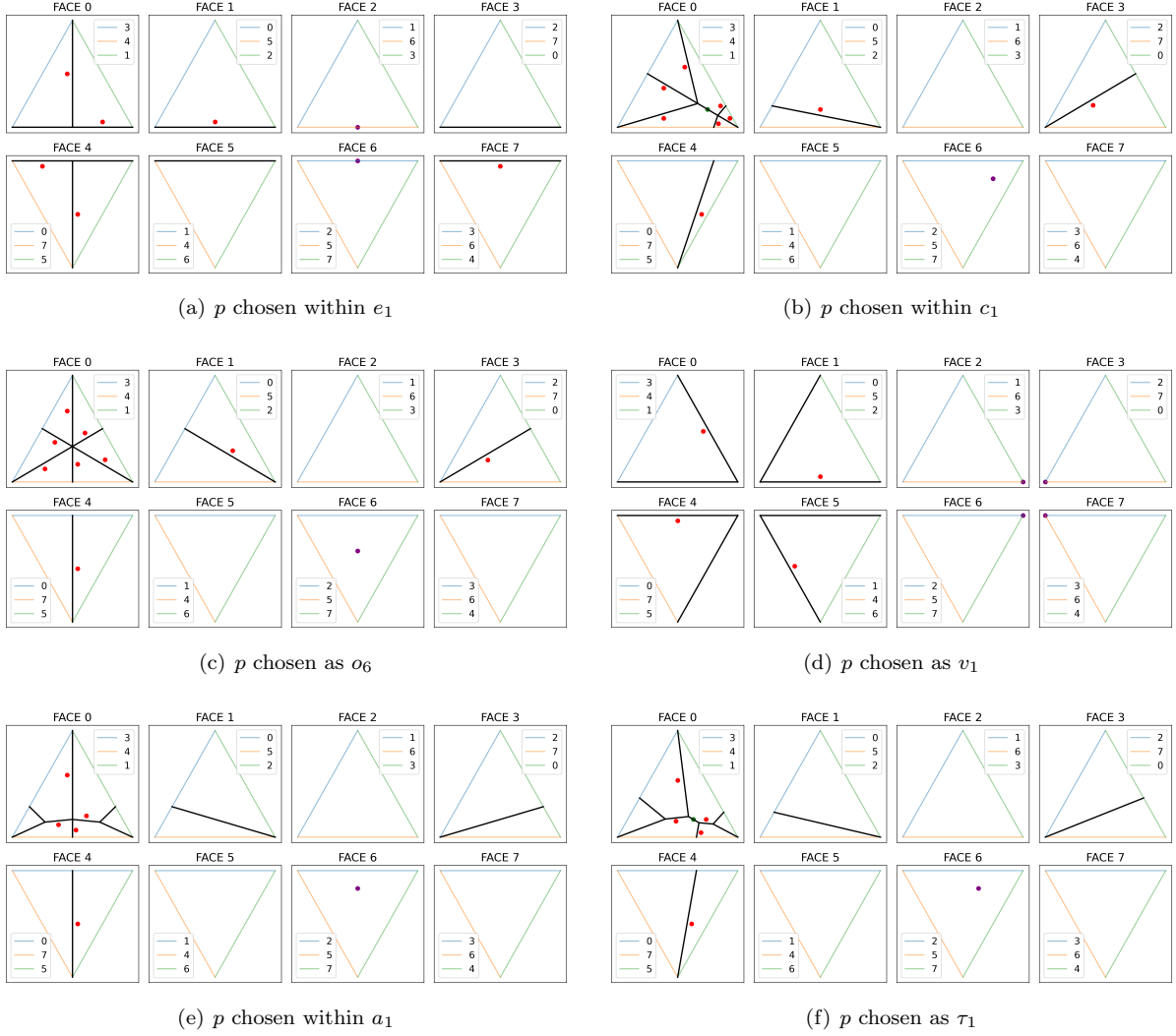


Figure 23: Geodesics chosen by our GMPR for a few start points (displayed as purple dots). Red dots indicate the direction from which the GMPR on E_2 approaches each cut locus line. Green dots indicate an arbitrary choice.

will prove continuity in a similar way. First, let A be a region of the octahedron with isomorphic cut locus. For a point on A , the cut locus vertices and line segments vary continuously. For the relative interior of a line segment, σ_2 is constructed to always approach from a consistent side, and thus always projects to a linear path in the same continuously varying Voronoi cell. As before, this implies that the GMPR is continuous, as we can use the path embedding associated with this Voronoi cell to pull each path in $\mathbb{R}^2$ back into X . Thus, σ_2 must be continuous with its first factor restricted to any region A of isomorphic cut locus.

Assume for the sake of contradiction that there exists a discontinuity at point (p, q) . We claim it must be witnessed by two distinct regions of isomorphic cut locus A_1 and A_2 , and sequence $(p_n, q_n)_n$ that converges to (p, q) such that $(p_n)_n \subseteq A_1$ and $\lim_{n \rightarrow \infty} p_n \in A_2$. It is certainly true that the discontinuity is witnessed by a sequence, as both $X \times X$ and GX are metric spaces. Then we may produce a sequence $(p_n, q_n)_n \subseteq E_2$ that approaches $(p, q) \in E_2$ such that each $\sigma_2(p_n, q_n)$ is ε -far from $\sigma_2(p, q)$ for fixed $\varepsilon > 0$. Since we partitioned the octahedron into finitely many regions of isomorphic cut locus, there is a region A_1 of isomorphic cut locus such that an infinite subsequence $(p_n)_{n \in M} \subseteq A_1$. Without loss of generality we may assume this is the whole sequence. Finally, $p \notin A_1$, since σ_2 is continuous when p is restricted to A_1 . We may additionally assume that for each $(p_n, q_n)_n$, q_n is always on the same cut locus line of p_n . We may do this since there are finitely many of these for p_n chosen in A_1 , and q_n must be on the relative interior of one of them. This indicates (in rough terms) that any discontinuity arises from a chosen direction for a line in the cut locus of points in A_1 not matching the chosen direction of the line's limit in A_2 . We will show that this cannot be true, as our construction ensures consistency of the choice of side for each line. Since the limit $p \in A_2$, we need only check regions A_1, A_2 such that $\overline{A_1} \cap A_2 \neq \emptyset$.

Assume $(p_n)_n \subseteq a_i$. There are four cut locus lines we must consider, and the limit p is either some o_j or the midpoint of an edge e_j . Since other a_i are symmetric, we assume $a_i = a_1$, $e_j = e_1$, and $o_j = o_6$. This case is visualized in Figures 13 and 23(e).

- We find from Algorithm 2 that if $(p_n)_n$ approaches o_6 , cut locus lines $\ell^{\{1,5\}}$ and $\ell^{\{2,4\}}$ both approach o_0 , the center of Face 0. Thus, if the $(q_n)_n$ are on either of these lines, then $q = o_0$. In this case, $(p, q) = (o_6, o_0) \in E_5$, and $(p_n, q_n)_n$ cannot witness a discontinuity in σ_2 . For cut locus lines $\ell^{\{1,2\}}$ and $\ell^{\{4,5\}}$, we find that their limits are the lines of the same name as $(p_n)_n$ approaches o_6 . Thus, if $(q_n)_n$ is on any of these lines, q must be on the closure of the limiting cut locus line. Additionally, as $(p_n)_n$ approaches o_6 , each line's endpoint on an octahedron vertex is fixed. From construction of σ_2 , this implies that the clockwise side of these cut locus lines is consistent with the clockwise side of their limits. Then if q is on the interior of a cut locus line, $(\sigma_2(p_n, q_n))_n$ must approach $\sigma_2(p, q)$, as the geodesics chosen are on a consistent side of a continuously varying cut locus line. If q is instead on an endpoint, $(p, q) \in E_5$, and $(p_n, q_n)_n$ cannot witness a discontinuity in σ_2 . Thus, $(p_n, q_n)_n$ cannot witness a discontinuity if $p = o_6$. In the analysis of other cases, we will not mention the case where q is the endpoint of a cut locus line, as this always results in $(p, q) \notin E_2$.
- Now assume $(p_n)_n$ approaches the midpoint of e_1 . We find cut locus lines $\ell^{\{1,2\}}$ and $\ell^{\{4,5\}}$ approach lines of the same name (in Figure 18) as $(p_n)_n$ approaches p , and their respective endpoints on octahedron vertices are fixed. Then as before, the clockwise side of these lines are consistent and $(p_n, q_n)_n$ cannot witness a discontinuity if $(q_n)_n$ is chosen on these lines. The cut locus lines $\ell^{\{1,5\}}$ and $\ell^{\{2,4\}}$ approach the lines of the same name as $(p_n)_n$ approaches e_1 . Our construction of σ_2 ensures that our chosen side for these cut locus lines is consistent with their limits at $p \in e_1$. Then, $(p_n, q_n)_n$ cannot witness a discontinuity if $(q_n)_n$ is chosen on these lines.

Therefore, there cannot be a discontinuity witnessed by $(p_n)_n \subseteq a_i$.

Assume $(p_n)_n \subseteq c_i$. There are seven cut locus lines we must consider, and the limit p is some v_j or o_j . Since other c_i are symmetric, we assume $c_i = c_1$, $v_j = v_1$, and $o_j = o_6$. This case is visualized in Figures 14 and 23(b).

- If $p = o_6$, we find that cut locus line $\ell^{\{1,4\}}$ approaches o_0 , the center point of Face 0. If $(q_n)_n$ is chosen on this line, $(p, q) = (o_6, o_0) \in E_5$, and $(p_n, q_n)_n$ cannot witness a discontinuity. The six other cut locus lines approach lines of the same name as $(p_n)_n$ approaches o_6 . Their respective endpoints on octahedron vertices are fixed, so the clockwise side of each cut locus line is consistent

with the clockwise side of their limits. Then, as before, $(p_n, q_n)_n$ cannot witness a discontinuity if $(q_n)_n$ is chosen on any of these lines.

- If $p = v_1$, we find as $(p_n)_n$ approaches v_1 , cut locus lines $\ell^{\{1,4\}}$ and $\ell^{\{0,5\}}$ both approach the antipode of v_1 (let this be v_4). Then if $(q_n)_n$ is chosen on these lines, $q = v_4$ and $(p, q) = (v_1, v_4) \in E_5$. The cut locus lines $\ell^{\{0,1\}}$, $\ell^{\{1,2\}}$, $\ell^{\{3,4\}}$, and $\ell^{\{4,5\}}$ approach the four octahedron edges that are incident to v_4 as $(p_n)_n$ approaches v_1 . Their respective endpoints on octahedron vertices are fixed, and their other endpoint becomes v_4 . Since we ignore v_4 when determining orientation on these cut locus lines, the clockwise side of each of these cut locus lines is consistent with the clockwise side of their limit. Thus, $(p_n, q_n)_n$ cannot witness a discontinuity if $(q_n)_n$ is chosen on any of these lines. Finally, as $(p_n)_n$ approaches v_1 , the cut locus line $\ell^{\{2,3\}}$ approaches a line across Faces 3 and 0 whose endpoints are v_1 and v_4 . However, this line does not exist on the cut locus of v_1 . Then if $(q_n)_n$ is on this line, $(p, q) \in E_5$ (if q is v_1 , v_4 , or the midpoint of an octahedron edge) or $(p, q) \in E_1$ (otherwise).

Therefore, there cannot be a discontinuity witnessed by $(p_n)_n \subseteq c_i$.

Assume $(p_n)_n \subseteq e_i$. There are four cut locus lines we must consider, and the limit p is one of two octahedron vertices v_j, v_k . Since other e_i are symmetric, we assume $e_i = e_1$ and $p = v_1$ (we need not check both endpoints of e_1). This case is visualized in Figures 17, 18(a), and 23(a).

As $(p_n)_n$ approaches v_1 , cut locus lines $\ell^{\{1,2\}}$, $\ell^{\{1,5\}}$, and $\ell^{\{4,5\}}$ approach three of the cut locus lines of v_1 . The endpoints of these lines on their respective octahedron vertices are fixed, and their other endpoint becomes v_4 (the antipode of v_1). Then as before, the clockwise side of each of these lines is consistent with their limit, and $(p_n, q_n)_n$ cannot witness a discontinuity if $(q_n)_n$ is chosen on any of these lines.

The remaining line $\ell^{\{2,4\}}$ approaches a line that starts at v_1 , ends at v_4 , consists of two octahedron edges, and includes the remaining cut locus line of v_1 (equivalently, this line extends the remaining cut locus line of v_1 by one more octahedron edge). For $p_n \in e_1$ we use the endpoint that does *not* approach v_4 to determine the clockwise side of this line. When $p = v_1$, we use the endpoint that is not v_4 to determine the clockwise side, and thus the part of this line that approaches the cut locus line of v_1 has a clockwise side consistent with its limit. Thus, if the limit q is on this cut locus line, $(p_n, q_n)_n$ cannot witness a discontinuity. Otherwise, (p, q) must either be in E_5 (if q is an octahedron vertex or a midpoint of any octahedron edge) or in E_1 (in any other case). Thus, there cannot be a discontinuity witnessed by $(p_n, q_n)_n \subseteq e_i$.

Assume $(p_n)_n \subseteq \tau_i$. There are five cut locus lines to consider, and p is on some o_j, v_j, e_j, a_j , or c_j . Since other τ_i are symmetric, we assume $\tau_i = \tau_1$, $v_j = v_1$, $e_j = e_1$, $a_j = a_1$, $c_j = c_1$, and $o_j = o_6$. This case is visualized in Figures 15 and 23(f).

- If $p = o_6$, cut locus lines $\ell^{\{1,4\}}$, $\ell^{\{1,5\}}$, and $\ell^{\{2,4\}}$ approach the antipode o_0 as $(p_n)_n$ approaches o_6 . Thus, $(q_n)_n$ chosen on any of these lines results in $q = o_0$ and $(p, q) \in E_5$. The two other cut locus lines approach lines of the same name as $(p_n)_n$ approaches o_6 . Their respective endpoints on octahedron vertices are fixed, so the clockwise side of each cut locus line is consistent with the clockwise side of their limits. Then, as before, $(p_n, q_n)_n$ cannot witness a discontinuity if $(q_n)_n$ is chosen on any of these lines.
- If $p = v_1$, cut locus lines $\ell^{\{1,4\}}$ and $\ell^{\{1,5\}}$ approach the antipode v_4 as $(p_n)_n$ approaches v_1 . If $(q_n)_n$ were chosen on any of these lines, $(p, q) = (v_1, v_4) \in E_5$. As $(p_n)_n$ approaches v_1 , the lines $\ell^{\{1,2\}}$ and $\ell^{\{4,5\}}$ approach cut locus lines of v_1 . The endpoints of these lines on their respective octahedron vertices are fixed, and each line's other endpoint approaches v_4 . Thus, the clockwise side of each cut locus line is consistent with the clockwise side of their limits, and $(p_n, q_n)_n$ cannot witness a discontinuity if $(q_n)_n$ is chosen on these lines. Finally, any $(q_n)_n$ chosen on cut locus line $\ell^{\{2,4\}}$ approaches the bottom edge of Face 0, which is a cut locus line of v_1 . From construction of σ_2 , the chosen side of $\ell^{\{2,4\}}$ is consistent with the clockwise

side of the bottom edge of Face 0. Then $(p_n, q_n)_n$ cannot witness a discontinuity if $(q_n)_n$ is chosen on this line.

- If $p \in e_1$, cut locus line $\ell^{\{1,4\}}$ approaches the vertex of the cut locus as $(p_n)_n$ approaches p . Then if $(q_n)_n$ is chosen on this line, q is this vertex, and $(p, q) \notin E_2$.
The lines $\ell^{\{1,2\}}$ and $\ell^{\{4,5\}}$ continuously approach the lines of the same name as $(p_n)_n$ approaches p , keeping their respective endpoints at an octahedron vertex constant. Then, as before, the clockwise sides of these lines are consistent with their limits, and $(p_n, q_n)_n$ cannot witness a discontinuity if $(q_n)_n$ is chosen on these lines.
For $(q_n)_n$ chosen on the two remaining lines, q must be on the bottom edge of Face 0, which is a cut locus line of p . Specifically, q must be on the cut locus line with the same name as those that $(q_n)_n$ are on. As before, from construction of σ_2 , the chosen side of each cut locus line is consistent with the clockwise side of the line q is on, and $(p_n, q_n)_n$ cannot witness a discontinuity if $(q_n)_n$ is chosen on these lines.
- If $p \in a_1$, the cut locus line $\ell^{\{1,4\}}$ approaches a cut locus vertex as $(p_n)_n$ approaches p . Thus, if $(q_n)_n$ were on this line, q is this vertex and $(p, q) \notin E_2$.
The lines $\ell^{\{1,2\}}$ and $\ell^{\{4,5\}}$ approach the lines of the same name in Figure 13, with their respective endpoints at octahedron vertices fixed. Then $(p_n, q_n)_n$ cannot witness a discontinuity with $(q_n)_n$ chosen on these lines.
The lines $\ell^{\{1,5\}}$ and $\ell^{\{2,4\}}$ approach the lines of the same name in Figure 13. From construction of σ_2 , our choice of direction to approach this line is consistent with its limit, so $(p_n, q_n)_n$ cannot witness a discontinuity with $(q_n)_n$ chosen on this line.
- If $p \in c_1$, the cut locus lines $\ell^{\{1,5\}}$ and $\ell^{\{2,4\}}$ approach vertices of the cut locus as $(p_n)_n$ approaches p . If $(q_n)_n$ were on either line, then q is a cut locus vertex and $(p, q) \notin E_2$.
The line $\ell^{\{1,4\}}$ continuously approaches the line of the same name as $(p_n)_n$ approaches p , and by construction of σ_2 , our choice of direction to approach this line is consistent with its limit, so $(p_n, q_n)_n$ cannot witness a discontinuity with $(q_n)_n$ chosen on this line.
The cut locus lines $\ell^{\{1,2\}}$ and $\ell^{\{4,5\}}$ approach the lines of the same name as $(p_n)_n$ approaches p , with their respective endpoints at octahedron vertices fixed. Thus, the clockwise side of these lines is consistent with their limits, and $(p_n, q_n)_n$ cannot witness a discontinuity with $(q_n)_n$ is chosen on these lines.

Therefore, there cannot be a discontinuity witnessed by $(p_n)_n \subseteq \tau_i$.

Finally, if $(p_n)_n \subseteq \{v_i\}$, then $p = v_i$, so $(p_n, q_n)_n$ cannot witness a discontinuity. Similarly, $(p_n)_n \subseteq \{o_i\}$ results in $p = o_i$.

Thus, $GC(X) \leq 4$, needing at most five sets in a minimal geodesic motion planner. With the lower bound in Section 3.3.2, this implies that the geodesic complexity of an octahedron is four, and a minimal geodesic motion planner on this space requires a partition of exactly five sets.

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A Tetrahedron

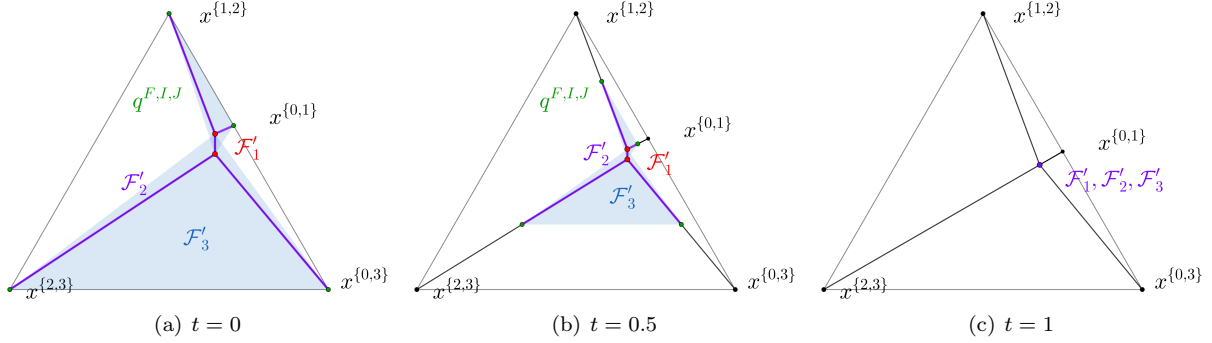


Figure 24: Visualization of $\mathcal{F}_i$ as p varies towards p^* along Face 3

We would like to visualize our method's application to the tetrahedron by displaying our sets $\mathcal{F}_i$ in a way that imitates Figure 4(a). Since some embedded simplices are 3-dimensional, we use the same method as in Figure 10 and plot a few slices of one dimension. For $i > 0$, each $F_i \in \mathcal{F}_i$ is a map $\Delta_i \rightarrow X \times X$. Recall from Definition 2.3 that $T_1 = [0, 1]$ and for $t \in T_1$, $D_{t,i} = \{x \in \Delta_i : x_0 = t\}$. By Lemma 2.1, we have that the image $\pi_2(D_{t,i})$ is the convex hull of i points. Let this hull be F'_i , and let $\mathcal{F}'_i$ be the union $\bigcup_{F_i \in \mathcal{F}_i} F'_i$. In

Figure 24, we plot $\mathcal{F}'_i$ for a few values of t for cases where the first factor embedding is p_3 .

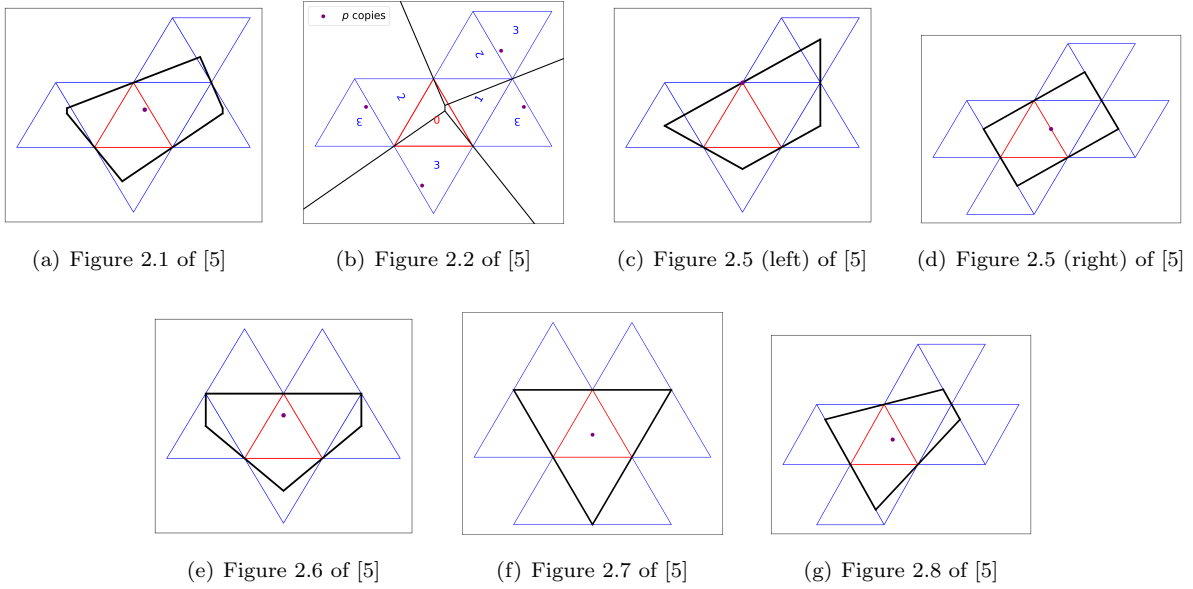


Figure 25: Recreation of figures in [5, Sec. 2] using Algorithm 2

B Octahedron

We will find the equations of the lines $\ell^{\{i,j\}}$ for choices of p on Face 6 of the octahedron. These equations are well defined for p on all of Face 6 other than its vertices.

- $\ell^{\{0,1\}}(p) := t \mapsto \left(\frac{p_1}{4} - \sqrt{3}\frac{p_2}{4} + 5\frac{\sqrt{3}}{2}, \sqrt{3}\frac{p_1}{4} + \frac{p_2}{4} + \frac{5}{2}\right) + \left(-\sqrt{3}\frac{p_1}{4} + 3\frac{p_2}{4} - \frac{3}{2}, -3\frac{p_1}{4} - \sqrt{3}\frac{p_2}{4} - \frac{\sqrt{3}}{2}\right)t.$
- $\ell^{\{0,2\}}(p) := t \mapsto \left(\frac{p_1}{4} + \sqrt{3}\frac{p_2}{4} + \frac{\sqrt{3}}{2}, -\sqrt{3}\frac{p_1}{4} + \frac{p_2}{4} + \frac{5}{2}\right) + \left(\sqrt{3}\frac{p_1}{4} + 3\frac{p_2}{4} - \frac{3}{2}, -3\frac{p_1}{4} + \sqrt{3}\frac{p_2}{4} - 5\frac{\sqrt{3}}{2}\right)t.$
- $\ell^{\{0,3\}}(p) := t \mapsto (p_1, p_2 + 1) + (0, -3\sqrt{3})t.$
- $\ell^{\{0,4\}}(p) := t \mapsto \left(\frac{p_1}{4} - \sqrt{3}\frac{p_2}{4} + \sqrt{3}, \sqrt{3}\frac{p_1}{4} + \frac{p_2}{4} - 2\right) + \left(-\sqrt{3}\frac{p_1}{4} + 3\frac{p_2}{4} + 3, -3\frac{p_1}{4} - \sqrt{3}\frac{p_2}{4} - 2\sqrt{3}\right)t.$
- $\ell^{\{0,5\}}(p) := t \mapsto \left(\frac{p_1}{4} + \sqrt{3}\frac{p_2}{4} + 2\sqrt{3}, -\sqrt{3}\frac{p_1}{4} + \frac{p_2}{4} - 2\right) + \left(\sqrt{3}\frac{p_1}{4} + 3\frac{p_2}{4} + 3, -3\frac{p_1}{4} + \sqrt{3}\frac{p_2}{4} - \sqrt{3}\right)t.$
- $\ell^{\{1,2\}}(p) := t \mapsto \left(\frac{-p_1}{2}, 4 - \frac{p_2}{2}\right) + \left(\sqrt{3}\frac{p_1}{2}, \sqrt{3}\frac{p_2-4}{2}\right)t.$
- $\ell^{\{1,3\}}(p) := t \mapsto \left(\frac{p_1}{4} - \sqrt{3}\frac{p_2}{4} - \frac{\sqrt{3}}{2}, \sqrt{3}\frac{p_1}{4} + \frac{p_2}{4} + \frac{5}{2}\right) + \left(\sqrt{3}\frac{p_1}{4} - 3\frac{p_2}{4} + \frac{3}{2}, 3\frac{p_1}{4} + \sqrt{3}\frac{p_2}{4} - 5\frac{\sqrt{3}}{2}\right)t.$
- $\ell^{\{1,4\}}(p) := t \mapsto \left(\frac{-p_1}{2} - \sqrt{3}\frac{p_2}{2} + \frac{\sqrt{3}}{2}, \sqrt{3}\frac{p_1}{2} - \frac{p_2}{2} - \frac{1}{2}\right) + \left(\frac{9}{2}, -3\frac{\sqrt{3}}{2}\right)t.$
- $\ell^{\{1,5\}}(p) := t \mapsto \left(\frac{-p_1}{2} + 3\frac{\sqrt{3}}{2}, \frac{-p_2}{2} - \frac{1}{2}\right) + \left(\sqrt{3}\frac{p_1}{2} + \frac{9}{2}, \sqrt{3}\frac{p_2-1}{2}\right)t.$
- $\ell^{\{2,3\}}(p) := t \mapsto \left(\frac{p_1}{4} + \sqrt{3}\frac{p_2}{4} - 5\frac{\sqrt{3}}{2}, -\sqrt{3}\frac{p_1}{4} + \frac{p_2}{4} + \frac{5}{2}\right) + \left(-\sqrt{3}\frac{p_1}{4} - 3\frac{p_2}{4} + \frac{3}{2}, 3\frac{p_1}{4} - \sqrt{3}\frac{p_2}{4} - \frac{\sqrt{3}}{2}\right)t.$
- $\ell^{\{2,4\}}(p) := t \mapsto \left(\frac{-p_1}{2} - 3\frac{\sqrt{3}}{2}, \frac{-p_2}{2} - \frac{1}{2}\right) + \left(-\sqrt{3}\frac{p_1}{2} + \frac{9}{2}, \sqrt{3}\frac{1-p_2}{2}\right)t.$
- $\ell^{\{2,5\}}(p) := t \mapsto \left(\frac{-p_1}{2} + \sqrt{3}\frac{p_2}{2} - \frac{\sqrt{3}}{2}, -\sqrt{3}\frac{p_1}{2} - \frac{p_2}{2} - \frac{1}{2}\right) + \left(\frac{9}{2}, 3\frac{\sqrt{3}}{2}\right)t.$
- $\ell^{\{3,4\}}(p) := t \mapsto \left(\frac{p_1}{4} - \sqrt{3}\frac{p_2}{4} - 2\sqrt{3}, \sqrt{3}\frac{p_1}{4} + \frac{p_2}{4} - 2\right) + \left(-\sqrt{3}\frac{p_1}{4} + 3\frac{p_2}{4} + 3, -3\frac{p_1}{4} - \sqrt{3}\frac{p_2}{4} + \sqrt{3}\right)t.$
- $\ell^{\{3,5\}}(p) := t \mapsto \left(\frac{p_1}{4} + \sqrt{3}\frac{p_2}{4} - \sqrt{3}, -\sqrt{3}\frac{p_1}{4} + \frac{p_2}{4} - 2\right) + \left(\sqrt{3}\frac{p_1}{4} + 3\frac{p_2}{4} + 3, -3\frac{p_1}{4} + \sqrt{3}\frac{p_2}{4} + 2\sqrt{3}\right)t.$
- $\ell^{\{4,5\}}(p) := t \mapsto \left(\frac{-p_1}{2}, \frac{-p_2}{2} - 5\right) + \left(\sqrt{3}\frac{p_1}{2}, \sqrt{3}\frac{p_2+2}{2}\right)t.$

For $j \equiv i + 1 \pmod{6}$, let $x^{\{i,j\}}(p)$ be the intersection of $\ell^{\{i,j\}}(p)$ with the boundary of Face 0. We find $x^{\{i,j\}}(p)$ for arbitrary p :

- $x^{\{0,1\}}(p) = \left(3\frac{\sqrt{3}p_1+p_2+2}{3p_1-\sqrt{3}p_2+4\sqrt{3}}, \frac{-3p_1-5\sqrt{3}p_2+2\sqrt{3}}{3p_1-\sqrt{3}p_2+4\sqrt{3}}\right).$
- $x^{\{1,2\}}(p) = (0, 2).$
- $x^{\{2,3\}}(p) = \left(3\frac{-\sqrt{3}p_1+p_2+2}{3p_1+\sqrt{3}p_2-4\sqrt{3}}, \frac{-3p_1+5\sqrt{3}p_2-2\sqrt{3}}{3p_1+\sqrt{3}p_2-4\sqrt{3}}\right).$
- $x^{\{3,4\}}(p) = (-\sqrt{3}, -1).$
- $x^{\{4,5\}}(p) = \left(\frac{3p_1}{p_2+2}, -1\right).$
- $x^{\{0,5\}}(p) = (\sqrt{3}, -1).$

From the intersections of lines $\ell^{\{i,j\}}$ and $\ell^{\{j,k\}}$, we calculate the $x^{(i,j,k)}$:

- $x^{\{0,1,2\}}(p) = \left(2p_1\frac{\sqrt{3}p_1+p_2+2}{p_1^2+2\sqrt{3}p_1+p_2^2-6p_2+8}, 2\frac{p_1^2+\sqrt{3}p_1p_2-2\sqrt{3}p_1+2p_2^2-8p_2}{p_1^2+2\sqrt{3}p_1+p_2^2-6p_2+8}\right).$
- $x^{\{0,1,3\}}(p) = \left(p_1, \frac{3p_1^2+\sqrt{3}p_1p_2-2\sqrt{3}p_1-12p_2}{\sqrt{3}p_1-3p_2+6}\right).$
- $x^{\{0,1,4\}}(p) = \left(\frac{p_1^2-2\sqrt{3}p_1p_2+6\sqrt{3}p_1+3p_2^2+6p_2}{4(p_1+\sqrt{3})}, \frac{\sqrt{3}p_1^2-2p_1p_2-2p_1-\sqrt{3}p_2^2-6\sqrt{3}p_2}{4(p_1+\sqrt{3})}\right).$
- $x^{\{0,1,5\}}(p) = \left(\frac{3p_1^2+2\sqrt{3}p_1p_2+10\sqrt{3}p_1+9p_2^2+18p_2}{\sqrt{3}p_1^2+12p_1+\sqrt{3}p_2^2+8\sqrt{3}}, \frac{-\sqrt{3}p_1^2-6p_1p_2-6p_1+\sqrt{3}p_2^2-10\sqrt{3}p_2}{\sqrt{3}p_1^2+12p_1+\sqrt{3}p_2^2+8\sqrt{3}}\right).$
- $x^{\{0,2,3\}}(p) = \left(p_1, \frac{-3p_1^2+\sqrt{3}p_1p_2-2\sqrt{3}p_1+12p_2}{\sqrt{3}p_1+3p_2-6}\right).$

$$\begin{aligned}
\bullet x^{\{0,2,4\}}(p) &= \left(\frac{3\sqrt{3}p_1^2+2p_1p_2-26p_1-3\sqrt{3}p_2^2-6\sqrt{3}p_2}{p_1^2+p_2^2-28}, \frac{-p_1^2+6\sqrt{3}p_1p_2-6\sqrt{3}p_1+p_2^2+26p_2}{p_1^2+p_2^2-28} \right). \\
\bullet x^{\{0,2,5\}}(p) &= \left(\frac{p_1^2+2\sqrt{3}p_1p_2+6\sqrt{3}p_1+3p_2^2+6p_2}{4(p_1+2\sqrt{3})}, \frac{-\sqrt{3}p_1^2-2p_1p_2-2p_1+\sqrt{3}p_2^2-6\sqrt{3}p_2}{4(p_1+2\sqrt{3})} \right). \\
\bullet x^{\{0,3,4\}}(p) &= \left(p_1, \frac{-3p_1^2-\sqrt{3}p_1p_2+2\sqrt{3}p_1-6p_2}{-\sqrt{3}p_1+3p_2+12} \right). \\
\bullet x^{\{0,3,5\}}(p) &= \left(p_1, \frac{-3p_1^2+\sqrt{3}p_1p_2-2\sqrt{3}p_1-6p_2}{\sqrt{3}p_1+3p_2+12} \right). \\
\bullet x^{\{0,4,5\}}(p) &= \left(2p_1 \frac{\sqrt{3}p_1+p_2+8}{p_1^2+2\sqrt{3}p_1+p_2^2+6p_2+8}, 2 \frac{-2p_1^2+\sqrt{3}p_1p_2-2\sqrt{3}p_1-p_2^2-2p_2}{p_1^2+2\sqrt{3}p_1+p_2^2+6p_2+8} \right). \\
\bullet x^{\{1,2,3\}}(p) &= \left(2p_1 \frac{-\sqrt{3}p_1+p_2+2}{p_1^2-2\sqrt{3}p_1+p_2^2-6p_2+8}, 2 \frac{p_1^2-\sqrt{3}p_1p_2+2\sqrt{3}p_1+2p_2^2-8p_2}{p_1^2-2\sqrt{3}p_1+p_2^2-6p_2+8} \right). \\
\bullet x^{\{1,2,4\}}(p) &= \left(p_1 \frac{p_1-\sqrt{3}p_2-2\sqrt{3}}{p_1+\sqrt{3}p_2-4\sqrt{3}}, \frac{p_1p_2-2p_1-\sqrt{3}p_2^2+4\sqrt{3}p_2}{p_1+\sqrt{3}p_2-4\sqrt{3}} \right). \\
\bullet x^{\{1,2,5\}}(p) &= \left(p_1 \frac{p_1+\sqrt{3}p_2+2\sqrt{3}}{p_1-\sqrt{3}p_2+4\sqrt{3}}, \frac{p_1p_2-2p_1+\sqrt{3}p_2^2-4\sqrt{3}p_2}{p_1-\sqrt{3}p_2+4\sqrt{3}} \right). \\
\bullet x^{\{1,3,4\}}(p) &= \left(\frac{p_1^2-2\sqrt{3}p_1p_2-6\sqrt{3}p_1+3p_2^2+6p_2}{4(p_1-2\sqrt{3})}, \frac{\sqrt{3}p_1^2-2p_1p_2-2p_1-\sqrt{3}p_2^2+6\sqrt{3}p_2}{4(p_1-2\sqrt{3})} \right). \\
\bullet x^{\{1,3,5\}}(p) &= \left(\frac{-3\sqrt{3}p_1^2+2p_1p_2-26p_1+3\sqrt{3}p_2^2+6\sqrt{3}p_2}{p_1^2+p_2^2-28}, \frac{-p_1^2-6\sqrt{3}p_1p_2+6\sqrt{3}p_1+p_2^2+26p_2}{p_1^2+p_2^2-28} \right). \\
\bullet x^{\{1,4,5\}}(p) &= \left(p_1 \frac{p_1-\sqrt{3}p_2+4\sqrt{3}}{p_1+\sqrt{3}p_2+2\sqrt{3}}, \frac{p_1p_2-2p_1-\sqrt{3}p_2^2-2\sqrt{3}p_2}{p_1+\sqrt{3}p_2+2\sqrt{3}} \right). \\
\bullet x^{\{2,3,4\}}(p) &= \left(\frac{-3p_1^2+2\sqrt{3}p_1p_2+10\sqrt{3}p_1-9p_2^2-18p_2}{\sqrt{3}p_1^2-12p_1+\sqrt{3}p_2^2+8\sqrt{3}}, \frac{-\sqrt{3}p_1^2+6p_1p_2+6p_1+\sqrt{3}p_2^2-10\sqrt{3}p_2}{\sqrt{3}p_1^2-12p_1+\sqrt{3}p_2^2+8\sqrt{3}} \right). \\
\bullet x^{\{2,3,5\}}(p) &= \left(\frac{p_1^2+2\sqrt{3}p_1p_2-6\sqrt{3}p_1+3p_2^2+6p_2}{4(p_1-\sqrt{3})}, \frac{-\sqrt{3}p_1^2-2p_1p_2-2p_1+\sqrt{3}p_2^2+6\sqrt{3}p_2}{4(p_1-\sqrt{3})} \right). \\
\bullet x^{\{2,4,5\}}(p) &= \left(p_1 \frac{-p_1-\sqrt{3}p_2+4\sqrt{3}}{-p_1+\sqrt{3}p_2+2\sqrt{3}}, \frac{-p_1p_2+2p_1-\sqrt{3}p_2^2-2\sqrt{3}p_2}{-p_1+\sqrt{3}p_2+2\sqrt{3}} \right). \\
\bullet x^{\{3,4,5\}}(p) &= \left(2p_1 \frac{-\sqrt{3}p_1+p_2+8}{p_1^2-2\sqrt{3}p_1+p_2^2+6p_2+8}, 2 \frac{-2p_1^2-\sqrt{3}p_1p_2+2\sqrt{3}p_1-p_2^2-2p_2}{p_1^2-2\sqrt{3}p_1+p_2^2+6p_2+8} \right).
\end{aligned}$$

C Future Work

We believe the following are true, in order of expected difficulty:

Conjecture C.1. *The geodesic complexity of an icosahedron is at least four (needing at least five sets).*

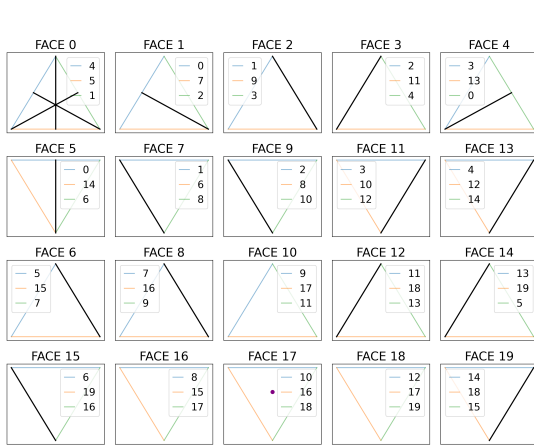
Conjecture C.2. *The geodesic complexity of a dodecahedron is at least four (needing at least five sets).*

Conjecture C.3. *The geodesic complexity of an icosahedron is at most four (needing at most five sets).*

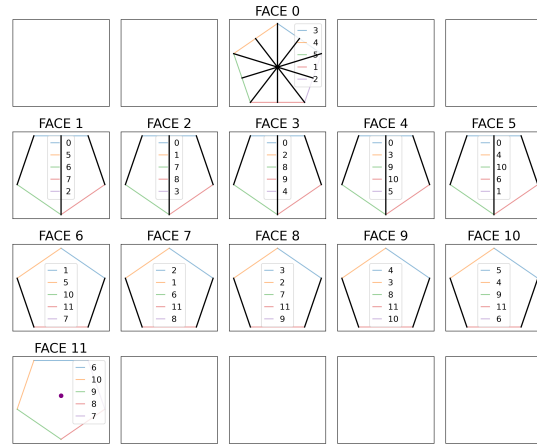
Conjecture C.4. *The geodesic complexity of a dodecahedron is at most four (needing at most five sets).*

Our motivation for believing Conjectures C.1 and C.2 is that the cut locus of the center of an icosahedron (resp. dodecahedron) face is a star with six (resp. ten) lines (Figure 26). We believe Theorem 2.1 can be used with $\mathcal{F}_i$ constructed similar to Section 3.3.2, and we expect Conjecture C.1 to be easier since the cut locus structure is more similar to the octahedron.

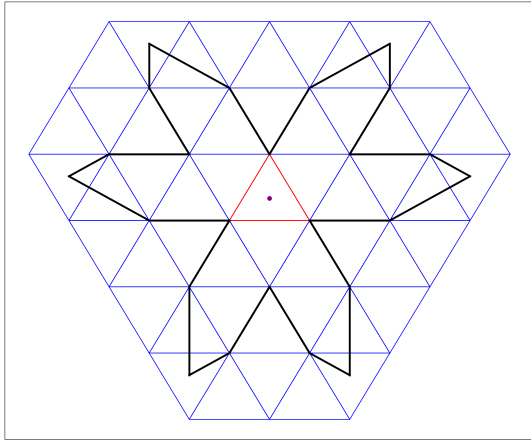
Our motivation for Conjectures C.3 and C.4 is that the tetrahedron [5], cube [6], and now the octahedron have geodesic complexity at most four. Increasing complexities of the cut loci on these spaces have required explicit geodesic motion planners to be made with more care. However, we do not notice differences with cut loci on the icosahedron and dodecahedron that would suggest this construction becomes impossible.



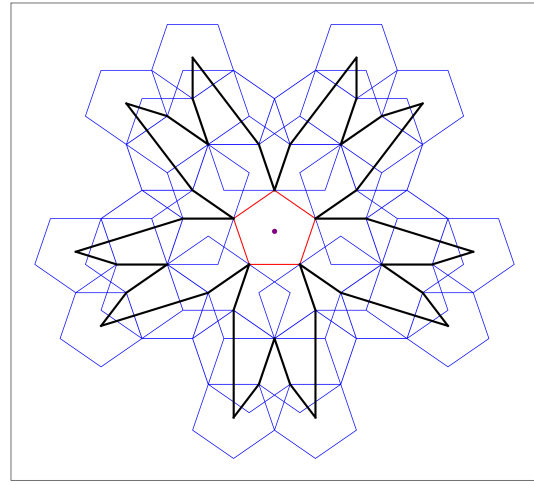
(a) Cut locus of point on icosahedron



(b) Cut locus of point on dodecahedron



(c) Voronoi star diagram of point on icosahedron



(d) Voronoi star diagram of point on dodecahedron

Figure 26: Output of Algorithm 2 on the center of an icosahedron and dodecahedron face