

## Geodesic complexity (it's actually quite simple)

**Topological complexity** [2] describes the difficulty of creating a continuous motion plan over a space  $X$ .  $X$  has topological complexity  $k$  (denoted  $TC(X) = k$ ) if  $k$  is the minimal number such that there is an open cover  $U_1 \cup U_2 \cup \dots \cup U_k = X \times X$  where each  $U_i$  has a local **motion planning rule**: a continuous map  $\sigma_i : U_i \rightarrow PX$  (with  $PX$  the space of paths in  $X$ ) such that  $\sigma(x, y)$  is a path from  $x$  to  $y$ .

**Geodesic complexity** [4] is analogous to topological complexity, with the caveat that all paths are **geodesics** (shortest paths with constant speed).  $GC(X) = k$  if  $k$  is the minimal number such that there is a **locally compact partition**  $E_0 \sqcup E_1 \sqcup \dots \sqcup E_k = X \times X$  where each  $E_i$  has a local **Geodesic Motion Planning Rule (GMPR)**  $\sigma_i : U_i \rightarrow GX$ . Here,  $GX \subseteq PX$  is the space of geodesics in  $X$ . We call  $\sigma_0, \dots, \sigma_k$  a **geodesic motion planner**.

### Theorem (simplex embedding theorem)

For integers  $0 \leq k_0 < k$ , if for each  $i \in \{k_0, k_0 + 1, \dots, k\}$  there exists a non-empty collection  $\mathcal{F}_i$  of embeddings into  $X \times X$  such that:

- Each  $\mathcal{F}_i$  consists of embeddings of  $i$ -dimensional simplices  $\Delta_i \hookrightarrow X \times X$ .
- For  $F \in \mathcal{F}_i$ , any geodesic in  $\pi_{GX}^{-1}(\text{relint}^*(F))$  extends to a GMPR on all of  $\text{img}(F)$ .
- With  $k_0 \leq i < k$ , for any  $F \in \mathcal{F}_i$  and GMPR  $\Gamma$  on  $\text{img}(F)$ , there is an  $F' \in \mathcal{F}_{i+1}$  such that:
  - The map  $F$  is the restriction of  $F'$  to a face of  $\Delta_{i+1}$ .
  - Each geodesic in  $\Gamma(\text{img}(F))$  is separated from any geodesics in  $\pi_{GX}^{-1}(\text{relint}^*(F'))$  by an open set.
  - There are a finite number of GMPRs on  $\text{img}(F')$ .

Then  $GC(X) \geq k - k_0$  (i.e. any geodesic motion planner requires  $\geq k - k_0 + 1$  sets).

Here,  $\text{relint}^*(F)$  is the **relative interior** of  $\text{img}(F)$ , and  $\pi_{GX} : GX \rightarrow X \times X$  maps a geodesic to its start and end point.

### Proof sketch

This method is inspired by arguments of Davis [1] and Recio-Mitter [4], whose proofs use sequences, sequences of sequences, etc. of elements in  $X \times X$ . Each iteration is constructed so it must be in a unique GMPR. Our theorem describes sufficient conditions for constructing these sequences.

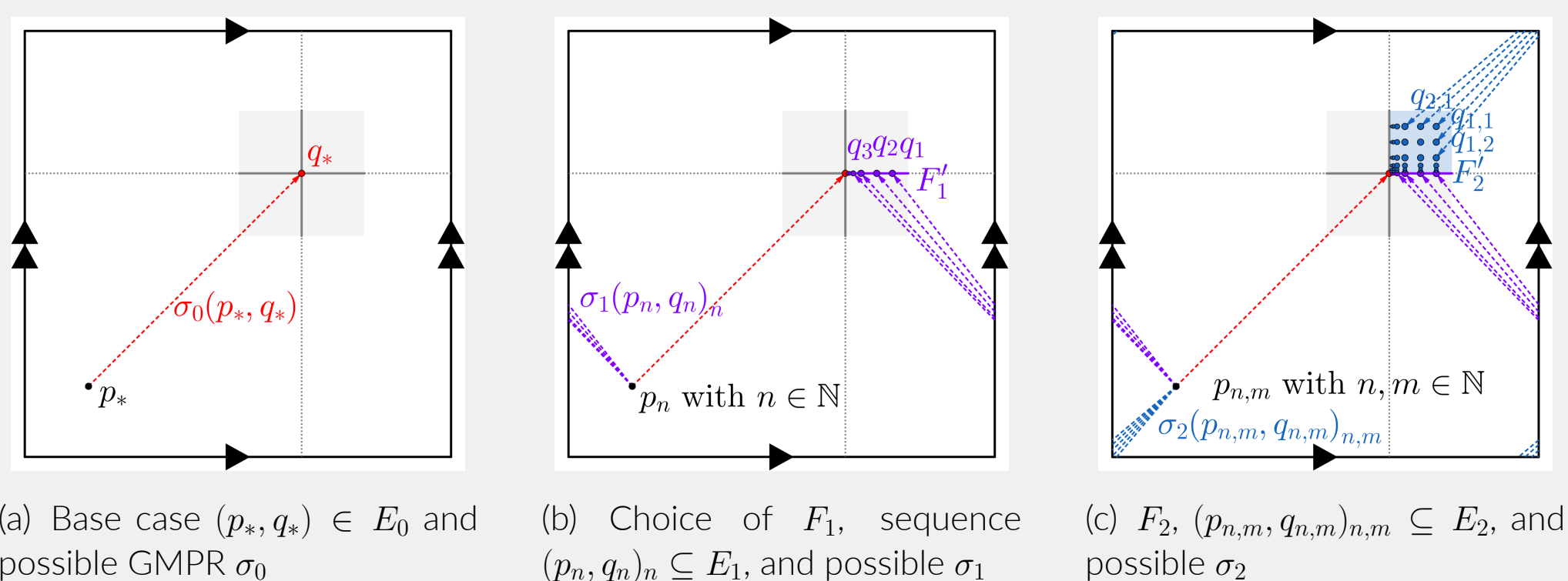


Figure 1. Mechanism of theorem when used on the 2-torus.

For  $X$  as the 2-torus, the sequence argument to show  $GC(X) \geq 2$  behaves as follows: Assume for contradiction there is a geodesic motion planner  $\sigma_0, \sigma_1$ . Let  $(p_*, q_*)$  be a point and its antipode, and assume  $(p_*, q_*) \in E_0$ . Construct  $(p_n, q_n)_n \subseteq E_k$  where  $p_i = p_*$ , and  $q_i$  approaches  $q_*$  from the side opposite  $\sigma_0(p_*, q_*)$  (as in Fig. 1(b)). Observe  $\sigma_0$  won't be continuous if  $(p_n, q_n)_n \subseteq E_0$ , so  $(p_n, q_n)_n \subseteq E_1$ . Repeat this process to produce  $(p_{n,m}, q_{n,m})_{n,m} \subseteq E_k$  (as in Fig. 1(c)), and observe for both  $k = 0$  and  $k = 1$ ,  $\sigma_k$  must be discontinuous.

To prove our theorem, we construct sequences on the images of appropriately selected simplex embeddings ( $\text{img}(F_1) = (p_*, F'_1)$ ,  $\text{img}(F_2) = (p_*, F'_2)$ ) are shown in Fig. 1(a,b)). The theorem conditions ensure that resulting sequences have the desired properties.

### Cut locus and its computation

The **cut locus** of a point  $p \in X$  is the set of all points in  $X$  with multiple distinct geodesics to  $p$ . The structure of the cut loci in  $X$  is vital to the computation of  $GC(X)$ , as it is exactly the set of points where a GMPR must make a decision.

For  $X$  the surface of a polyhedron, there exists a polynomial-time algorithm to compute the cut locus of a given point [3]. Using this algorithm, we obtain the result that the geodesic complexity of the octahedron is four.

## Lower bound on $GC(\text{octahedron})$

In this project, we inspect the surface of the octahedron, with the distance between points as the length of the shortest path between them. To show  $GC(\text{octahedron}) \geq 4$ , we use our simplex embedding theorem on embeddings around a particular point of the cut locus. We construct collections of  $(0, \dots, 4)$ -dimensional simplex embeddings to show this lower bound (this is the the strongest bound possible with this method). The cut loci relevant to our embeddings are those in Figure 2(c,d,e).

## Upper bound on $GC(\text{octahedron})$

We show  $GC(\text{octahedron}) \leq 4$  by constructing an explicit geodesic motion planner consisting of 5 GMPRs. We partition the start and end points mainly based on **multiplicity**, the number of unique geodesics between the pair (this strategy has been used for upper bounds on  $GC(\text{tetrahedron})$  and  $GC(\text{cube})$ ).

- $E_5$  is a discrete set containing all  $(p, q)$  where  $p$  and  $q$  are octahedron vertices, or the midpoint of an edge or face (this contains all  $(p, q)$  with multiplicity 6, and there exist no points with multiplicity 5).
- $E_1$  (resp.  $E_4$ ) contains all points not in  $E_5$  with multiplicity 1 (resp. 4).
- $E_3$  contains all points not in  $E_5$  with multiplicity 3 along with some with multiplicity 2 (these were added to ensure continuity of  $\sigma_2$ ).
- $E_2$  contains the remaining points with multiplicity 2.

For most of these sets, the associated  $\sigma_i$  is easy to construct. For  $\sigma_1$ , there is only one choice (since any  $(p, q) \in E_1$  has a unique geodesic).  $E_3, E_4$ , and  $E_5$  can be partitioned into small clopen sets, so  $\sigma_3, \sigma_4$ , and  $\sigma_5$  are easily constructed as piecewise functions. The most difficult GMPR is  $\sigma_2$ . To construct  $\sigma_2(p, q)$ , we consider the cut locus line that  $q$  lies on and approach from a counterclockwise direction, when applicable.

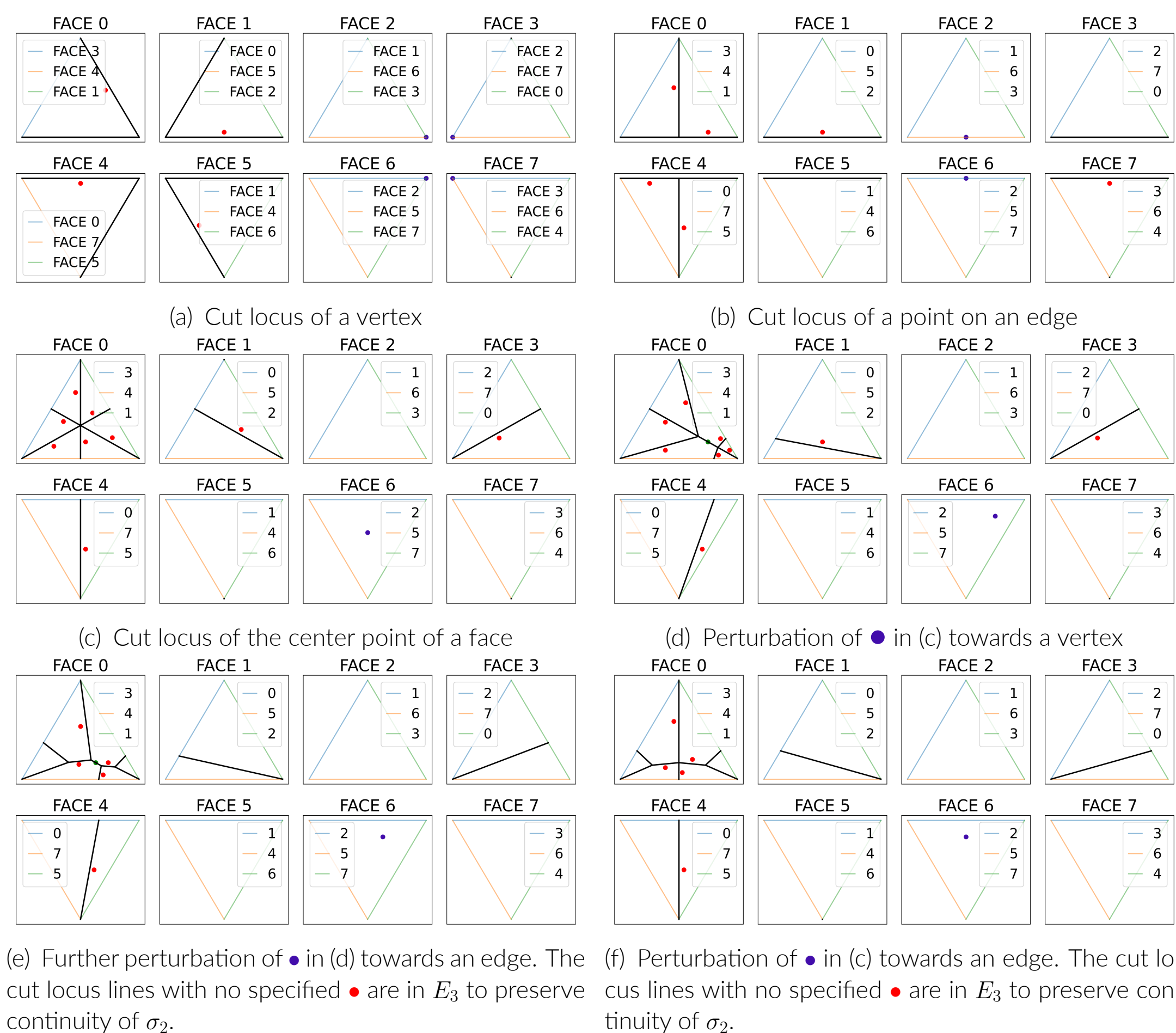


Figure 2. Construction of GMPR  $\sigma_2$ . A path starting from  $\bullet$  approaches a cut locus line from the side indicated by  $\bullet$ .  $\bullet$  indicates free choice on which side to approach from. Observe that when a cut locus line has a unique endpoint on an octahedron vertex,  $\sigma_2$  approaches the line from the 'right' side of that vertex (i.e. the first side encountered when moving counterclockwise around that vertex).

## References

- Donald M. Davis. Geodesic complexity of a tetrahedron. *The Graduate Journal of Mathematics*, 8(2):15–22, 2023.
- Michael Farber. Topological Complexity of Motion Planning. *Discrete and Computational Geometry*, 29:211–221, 2003.
- Joseph S. B. Mitchell, David M. Mount, and Christos H. Papadimitriou. The discrete geodesic problem. *SIAM Journal on Computing*, 16(4):647–668, 1987.
- David Recio-Mitter. Geodesic complexity of motion planning. *Journal of Applied and Computational Topology*, 5(1):141–178, 2021.